R.4 Polynomials in one variable

A ***monomial***: an algebraic expression of the form ax^n, where a is a real number, x is a variable and n is a nonnegative integer. *Example:* $2x^3$, $\sqrt{3}x^5$, 7

A ***binomial*** is the sum (or difference) of two monomials. *Example:* $2x^3 - \sqrt{3}x^5$

A ***trinomial*** is the sum (and/or difference) of three monomials. *Example:* $2x^3 - \sqrt{3}\,x^5 + 7$

A **polynomial** is the sum and/or difference of many monomials. It can be written in general as

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

The numbers $a_n, a_{n-1}, \ldots, a_0$ are called the ***coefficients*** of the polynomial. The highest power of x is called the ***degree of the polynomial*** and the coefficient of the monomial with the highest degree is called ***the leading coefficient***.

We say that a polynomial is in the standard form if it is written in the order of decreasing exponents of x

Example: $-2x^3 - 4x + 5$ is a polynomial of degree 3 and the leading coefficient is -2.

$\frac{3x-1}{x+1}$, $\sqrt{x+1}$ are not polynomials (not sums of monomials)

The like terms are monomials with the same variable raised to the same power. *Example:* $2x^5$ and $\sqrt{3}x^5$ are like terms but $2x^3$ and $\sqrt{3}x^5$ are not.

Operations on polynomials:

Addition/subtractions

(a) Use the distributive property to eliminate any parentheses
(b) Identify and add/subtract the like terms (add/subtract the coefficients of the like terms)
(c) Write the polynomial in the standard form

Example

$$3(2x^3 - 4x^2 - x + 3) - (-4x^2 + 2x - 5) =$$
$$\underline{6x^3} - \underline{\underline{12x^2}} - 3x + 9 + \underline{\underline{4x^2}} - 2x + 5 =$$
$$6x^3 + (-12x^2 + 4x^2) + (-3x - 2x) + (9 + 5) =$$
$$6x^3 + (-12 + 4)x^2 + (-3 - 2)x + 14 =$$
$$6x^3 - 8x^2 - 5x + 14$$

Multiplication

To multiply two monomials with the same variable, multiply their coefficients and raise the variable to the sum of their exponents. *Example*: $2x^2 \cdot (-3x^3) = (2)(-3)x^{2+3} = -6x^5$

Two multiply two polynomials

a) multiply each term of one polynomial by each term of the other polynomial
b) identify and combine like terms
c) Write the polynomial in the standard form

Example

$(2x^2 + 3)(-3x^3 + 4x - 5) = (2x^2)(-3x^3) + (2x^2)(4x) + (2x^2)(-5) + (3)(-3x^3) + (3)(4x) + (3)(-5) =$

$-6x^5 + 8x^3 - 10x^2 - 9x^3 + 12x - 15 =$

$-6x^5 - x^3 - 10x^2 + 12x - 15$

Special multiplication **formulas** (must be memorized)

(*) $(a - b)(a + b) = a^2 - b^2$
(**) $(a + b)^2 = a^2 + 2ab + b^2$
(***) $(a - b)^2 = a^2 - 2ab + a^2$

Example

a) $(2x-1)(2x+1) = (2x)^2 - 1^2 = 4x^2 - 1$
b) $(x+3)^2 = x^2 + 2\cdot 3\cdot x + 3^2 = x^2 + 6x + 9$

Long division of two polynomials

To divide two monomials with the same variable, divide the coefficients and subtract the exponents of the variable.

Example: $\frac{6x^3}{3x} = \frac{6}{3}x^{3-1} = 2x^2$

Example: Divide $6x^3 + 7x^2 + 3$ by $3x - 1$

Remark: $6x^3 + 7x^2 + 3$ is called the **dividend** and 3x-1 is called the **divisor**. You can perform division only when the degree of the dividend is greater or equal to the degree of the divisor.

1. Write the problem as a long division problem. Write both polynomials in the standard form. Include the missing terms, if any, with the coefficient zero

$\underline{2x^2 + 4x + 3}$	*this is the* ***quotient***
$3x - 1 \,)\, 6x^3 + 10x^2 + 5x + 3$	divide $6x^3$ by $3x$
$\underline{6x^3 - 2x^2}$	multiply $2x^2(3x-1)$ and subtract;
$12x^2 + 5x + 3$	divide $12x^2$ by $3x$
$\underline{12x^2 - 4x}$	multiply $4x(3x-1)$ and subtract;
$9x + 3$	divide 9x by 3x
$\underline{9x - 3}$	multiply 3(3x-1) and subtract
6	*this is the* ***remainder***

2. Divide the first term of the dividend ($6x^3$) by the first term of the divisor (3x)

$$\frac{6x^3}{3x} = 2x^2$$

and write it above the line

3. Multiply $2x^2$ by the divisor (3x-1) and write it beneath the dividend
$2x^2(3x-1) = 6x^3 - 2x^2$
4. Subtract: $6x^3 + 10x^2 + 5x + 3 - (6x^3 - 2x^2) = 6x^3 + 10x^2 + 5x + 3 - 6x^3 + 2x^2 = 12x^2 + 5x + 3$
5. Repeat the steps 2 – 4 with $12x^2 + 5x + 3$ as the dividend
6. Divide the first term ($12x^2$) by the first term of the divisor (3x)

$$\frac{12x^2}{3x} = 4x$$

And write it above the line

7. Multiply 4x by the divisor (3x-1): $4x(3x-1) = 12x^2 - 4x$
8. Subtract $(12x^2 + 5x + 3) - (12x^2 - 4x) = 12x^2 + 5x + 3 - 12x^2 + 4x = 9x + 3$
9. Repeat steps 2- 4 with 9x+3 as the dividend
$\frac{9x}{3x} = 3$
Multiply $3(3x-1) = 9x - 3$
Subtract $(9x + 3) - (9x-3) = 9x + 3 - 9x + 3 = 6$
10. STOP when the degree of the polynomial that plays the role of the dividend is less than the degree of the divisor

Note that

$$dividend = (quotient)(divisor) + remainder$$

or

$$\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$$

In the example above we can write

$6x^3+10x+5x+3 = (3x-1)(2x^2+ 4x + 3)+ 6$

Or

$$\frac{6x^3+10x^2+5x+3}{3x-1} = 2x^2+4x+3+\frac{6}{3x-1}$$

R.5 Factoring polynomials

In the product $3(2x-1)(3x^2+x+2)$, the polynomials 3, $(2x-1)$, and $(3x^2+x+2)$ are factors.

To factor a polynomial means to write it as a product of two or more polynomials

How to factor a polynomial:

(i) Factor out common factors:

- Identify common factor (if any) in all terms of the polynomial
- Use the distributive property: $ca + cb = c(a+b)$ to write the polynomial as a product

Example: a) $6x^3 + 2x^2 = 2\cdot 3\cdot x^2\cdot x + 2\cdot x^2 = 2x^2(3x+1)$

b) $(x+3)(x-2) + (x-2)x = (x+3)(x-2) + (x-2)x = (x-2)[(x+3)+x] = (x-2)(2x+3)$

c) $5(2x+1)^2 + (5x-6)\cdot 2(2x+1)\cdot 2 = 5(2x+1)^2 + (5x-6)\cdot 2(2x+1)\cdot 2$

$= (2x+1)[5(2x+1) + 4(5x-6)]$

$= (2x+1)[10x+5+20x-24] = (2x+1)(30x-19)$

(ii) Use a formula

- $a^2 - b^2 = (a-b)(a+b)$
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

Example: a) $4x^2 - 25 = (2x)^2 - (5)^2 = (2x-5)(2x+5)$

b) $x^3 + 8 = (x)^3 + (2)^3 = (x+2)(x^2-2x+4)$

c) $27 - 64x^3 = (3)^3 - (4x)^3 = (3-4x)(3^2 + 3(4x) + (4x)^2) = (3-4x)(9+12x+16x^2)$

(iii) factor by grouping

- group the polynomial into two (or more groups)
- factor each group
- if there is a common factor for each group, factor it out and factor the remaining polynomial. If there is no common factor for all the groups, group the original polynomial differently and try again.

Example:

a) $3x^2 + 6x - x - 2 = (3x^2+6x) + (-x-2) = 3x(x+2) + (-1)(x+2) = (x+2)(3x-1)$

b) $x^4 + x^3 + x + 1 = (x^4+x^3) + (x+1) = x^3(x+1) + (x+1) = (x+1)(x^3+1)$

$= (x+1))(x^3+1^3) = (x+1)(x+1)(x^2-x+1) = (x+1)^2(x^2-x+1)$

(iv) Factoring a trinomial $ax^2 + bx + c$, where a, b, c are integers

(iva) Factoring a trinomial $x^2 + bx + c$

- Find two numbers p, q such that

  $p\cdot q = c$ and $p + q = b$
- If such two numbers exist, then

  $x^2 + bx + c = (x+p)(x+q)$

Remarks: a) if c is negative, then p and q have opposite signs;

b) if c is positive then p, q have the same signs and if b is negative, then p,q are negative and if b is positive then p,q are positive

Example: Factor $x^2 + 5x + 6$.

We look for two numbers p and q such that $p\cdot q = 6$ and $p + q = 5$. Those numbers are 2 and 3. Therefore, $x^2 + 5x + 6 = (x+2)(x+3)$

(ivb) Factoring a trinomial $ax^2 + bx + c$, $a \neq 1$, a,b,c have no common factors

- Find two numbers p and q so that

  $p\cdot q = ac$ and $p + q = b$
- Rewrite the trinomial as

  $ax^2 + bx + c = ax^2 + px + qx + b$
- Factor by grouping

Example: Factor $-6z^2 + z + 1$

We look for two numbers p, q such that $p\cdot q = -6\cdot 1$ and $p + q = 1$.

Those numbers are -2 and 3.

Therefore

$-6z^2 + z + 1 = -6z^2 - 2z + 3z + 1 = (-6z^2 - 2z) + (3z+1) =$

$-2z(3z+1) + (3z+1) = (3z+1)(-2z+1)$

(v) Combine the above methods

Remark : - If a polynomial with integer coefficients cannot be factored we say it is **prime over the integers**

- A polynomial must be factored completely, which means that each factor must be prime

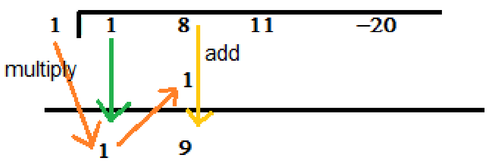
R.6 Synthetic division

Synthetic division is used to divide a polynomial P(x) by a binomial of the form x –c. It is a simplified form of long division.
The best way to explain this process is through an example.

Example: Divide $x^3 + 8x^2 + 11\,x – 20$ by x-1

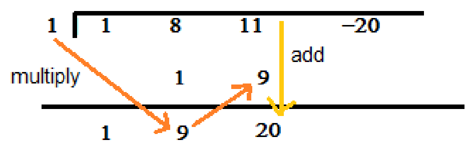
We start with writing the dividend in standard form (decreasing powers of x) and writing the coefficients in a row, remembering to insert a 0 when a power of x is missing.
Let $P(x) = x^3 + 8x^2 + 11\,x – 20$. The coefficients of P(x) are: 1, 8, 11, -20
Place the number 1 (c in the divisor x-c) to the left of the coefficients; use the division symbols

Step 1



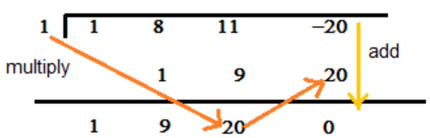
Write the leading coefficient(1) down below the horizontal bar. Multiply c =1, by the leading coefficient (1) and write the answer below the second coefficient (8). Add the numbers in that column and write the answer (9) below the horizontal bar.

Step 2

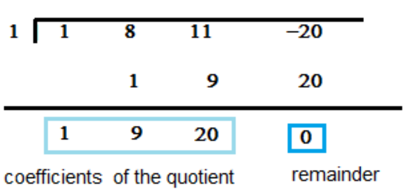


Repeat the process. Multiply c = 1 by the second entry(9) and write the answer(9) below the third coefficient. Add entries in that column and write the answer (20) below the horizontal bar.

Step 3



Step 4 The entries in the row below the horizontal bar contain the coefficients of the quotient (in the decreasing order of exponents. The last entry is the remainder.



The quotient, Q(x), is $\mathbf{Q(x) = x^2 + 9x + 20}$ and the remainder is 0.

Therefore, $P(x) = x^3 + 8x^2 + 11x -20 = (x-1)(x^2+ 9x+20)$.

Example

Use synthetic division to divide $3x^6 + 82x^3 + 27$ by $x + 2$.

Coefficients of the dividend are 3, 0, 0, 82, 0, 0 27 and c = -2

-2	3	0	0	82	0	0	27
		-6	12	-24	-116	232	-464
	3	-6	12	58	-116	232	-437

Coefficients of the quotient — Remainder

Therefore, $3x^6 + 82x^3 + 27 = (x+2)(3x^5 - x^4 + 12x^3 + 58x^2 - 116x + 232) - 437$

Example

Use synthetic division to determine whether $x + 3$ is a factor of $-4x^3 + 5x^2 + 8$

$x + 3$ will be a factor of $-4x^3 + 5x^2 + 8$ if the remainder is zero.

Coefficients of $-4x^3 + 5x^2 + 8$ are -4, 5, 0, 8 and c = -3

-3	- 4	5	0	8
		12	-51	153
	-4	17	-51	161

Coefficients of the quotient — Remainder

Since the remainder is not 0, x+3 is NOT a factor of $-4x^3 + 5x^2 + 8$.