8.1 & 8.2 Systems of Linear equations: Substitution and Elimination

Systems of two linear equations in two variables

A system of equations is a collection of two or more equations. A solution of a system in two variables is a pair of numbers that satisfies **all** the equations in the system.

Example: $\begin{cases} 3x + y = 4 \\ x + y = 2 \end{cases}$ is a system whose solution is the pair (1,1)

There are two types of systems: **consistent** (have a solution) and **inconsistent** (do not have a solution) Consistent system can be **dependent** (have infinitely many solutions) and **independent** (have only one solution)

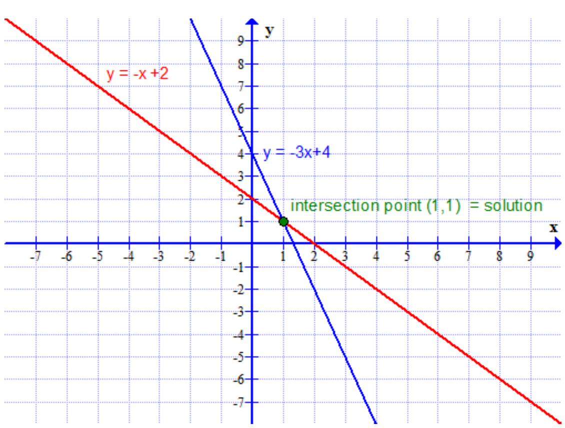
System of two linear equations with two variables can be represented by two straight lines.
In the system above one line is represented by the equation 3x + y = 4 and the other line by the equation x + y = 2
If the **lines intersect**, the point of intersection is the solution of the system. The system is consistent independent.
If the **lines are identical**, then there are infinitely many solutions. Each point on the line is a solution. The system is consistent dependent.
If the lines are **parallel**, there is no solution. The system is inconsistent.

System of equations can be solved graphically (you need a very accurate graph for that), by substitution , by elimination (addition), using matrices (we will not study this method) and using Cramer's Rule (we'll learn it in sec 12.3).

Graphical method: Carefully graph both equations in the same coordinate plane. Use graph paper and a ruler. The intersection of the two graphs, if any, is the solution.

Example: Solve by graphing $\begin{cases} 3x + y = 4 \\ x + y = 2 \end{cases}$

Line 3x + y = 4
y = -3x + 4 slope m = -3, y –intercept : (0,4)
Line x + y = 2
y = -x + 2 slope m = -1, y-intercept : (0,2)



Substitution method: Solve one of the equations for one of the variables, say y, substitute to the other equation and solve for x. Then, go back to the first equation to find value of y.

Example: Solve

$$\begin{cases} 2x + y = 4 \\ 3x - 4y = 6 \end{cases}$$

$$\begin{cases} y = 4 - 2x \\ 3x - 4(4 - 2x) = 6 \end{cases}$$

$$3x - 16 + 8x = 6$$

$$11x = 22$$

$$x = 2$$

$$\textit{substitute}$$

$$2(2) + y = 4$$

$$y = 0$$

$$\textit{solution}: (2,0)$$

Elimination (addition) method: Multiply one or both equations by suitable numbers, so that after the multiplication is done, coefficients of one of the variables, say x, are opposite numbers. Add both equations side by side to eliminate that variable. Solve the resulting equation for y. Substitute y in one of the equations and solve for x.

Example: (we eliminate y)

$$\begin{cases} 2x + y = 4 \\ 3x - 4y = 6 \end{cases}$$

Multiply first equation by 4, so the coefficients of y are 4 and -4

$$\begin{cases} 4 \cdot 2x + 4 \cdot y = 4 \cdot 4 \\ 3x \quad - 4y \ = 6 \end{cases}$$

Add side by side

$$11x + 0y = 22$$

Solve for x $x = 2$

Substitute x = 2 into the first equation

$$2(2) + y = 4$$

Solve for y $y = 0$

Solution: (2, 0)

Remark:

- if after substitution or elimination you obtain an equation that is always true (like 0 = 0), then the system has infinitely many solutions. The line represented by either equation contains all solutions
- If after substitution or elimination you obtain an equation that is always false (like 0 = 5), then the system has no solutions.

Systems of three linear equations with three variables

Example:

$$\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$$

Solution of such a system is a triple of numbers (a,b,c) that when substituted for x, y, z respectively, in each equation, makes each equation a true statement.
A system of three linear equation can have one solution, infinitely many solutions and no solutions

System of three equations with three variables can be solved by substitution and elimination.

Substitution method
Example: Solve

$$\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$$

1) Solve one of the equations for one of the variables (choose an equation that is easiest to solve for one of the variables).
First equation can be easily solved for x: x = 6 - y+ z
2) Substitute x in the remaining two equations with the expression obtained in step 1)

$$\begin{cases} x = 6 - y + z \\ 3(6 - y + z) - 2y + z = -5 \\ (6 - y + z) + 3y - 2z = 14 \end{cases}$$

3) Simplify the two equations. Notice that the last two equations contain only y and z.

$$\begin{cases} x = 6 - y + z \\ 18 - 3y + 3z - 2y + z = -5 \\ 6 - y + z + 3y - 2z = 14 \end{cases}$$

$$\begin{cases} x = 6 - y + z \\ -5y + 4z = -23 \\ 2y - z = 8 \end{cases}$$

4) Solve the resulting system of two equations with two variables (y and z)

$$\begin{cases} -5y + 4z = -23 \\ 2y - z = 8 \end{cases}$$

The solution is y = 3, z = -2
5) use the first equation to find value of x:
x = 6 – y + z = 6 – 3 - 2 = 1
6) The solution is (1, 3, -2)

Elimination method

Example: Solve the system

$$\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$$

The idea is to eliminate one of the variables by multiplying the equations by suitable constants, adding two equations side by side and reducing the system to a system with two equations and two variables. We'll eliminate x .

1) Multiply the first equation by (-2) and add to the second equation

$$\begin{array}{r} -2x + 2y - 2z = 8 \\ \underline{2x - 3y + 4z = -15} \\ -y + 2z = -7 \end{array}$$

(this is now the equation 2)

2) Multiply the first equation by (-5) and add to the third equation

$$\begin{array}{r} -5x + 5y - 5z = 20 \\ \underline{5x + y - 2z = 12} \\ 6y - 7z = 32 \end{array}$$

(this is now the equation 3)

3) Solve the system

$$\begin{cases} -y + 2z = -7 \\ 6y - 7z = 32 \end{cases}$$

Using elimination we get y = 3, z = -2

4) Use the first equation to find x

x -3 - 2 = - 4

x = 1

5) solution is (1,3,-2)

Example: (Infinitely many solutions) Solve

$$\begin{cases} x - y + 5z = -2 \\ 2x + y + 4z = 2 \\ 2x + 4y - 2z = 8 \end{cases}$$

Using the elimination method (eliminate x using the first equation) we get the system

$$\begin{cases} x - y + 5z = -2 \\ 3y - 6z = 6 \\ 6y - 12z = 12 \end{cases}$$

Solving, by elimination, the system

$$\begin{cases} 3y - 6z = 6 \\ 6y - 12z = 12 \end{cases}$$

leads to the equation 0 = 0. This means that there are infinitely many solutions. To find the solution set, notice that the system reduces to

$$\begin{cases} x - y + 5z = -2 \\ 3y - 6z = 6 \end{cases} \quad \text{or} \quad \begin{cases} x - y + 5z = -2 \\ y - 2z = 2 \end{cases}.$$

We get y = 2 + 2z and x = y – 5z -2 = (2 +2z) -5z -2 = -3z.

Therefore, if z is any real number, then x = -3z, y = 2z+ 2, z = z is a solution. The solution set is {(x,y,z)| x = -3z, y = 2z+2, z is any real number}

9.5 System of linear equations: determinants and Cramer's Rule

Consider a 2x2 system of equations

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

Let's use the Elimination method to solve this general system. Multiplying the first equation by d and the second equation by (-b) gives

$$\begin{cases} adx + bdy = sd \\ (-b)cx + (-b)dy = (-b)t \end{cases}$$

Adding the equations side by side eliminates the y variable and gives the equation for x

$(ad - bc)x = sd - bt$

When $ad - bc \neq 0$, we can solve that equation for x: $x = \dfrac{sd - bt}{ad - bc}$. If $ad - bc = 0$ then this last equation has either no solution (when $sd - bt \neq 0$) or infinitely many solutions (x can be any real number when $sd - bt = 0$)

If $ad - bc \neq 0$, then (plugging in x to the first equation) we get, $y = \dfrac{at - sc}{ad - bc}$ and hence, the solution of the system is $\left(\dfrac{sd - bt}{ad - bc}, \dfrac{at - sc}{ad - bc}\right)$

Note that the x and y values of the solution (if it exists) have a very similar structure.

Definition:

A 2 by 2 **determinant**, denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Evaluate $\begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix}$

$$\begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix} = (-1) \cdot 4 - (-3) \cdot 2 = -4 + 6 = 2 =$$

Note that $\begin{vmatrix} s & b \\ t & d \end{vmatrix} = sd - bt$ and $\begin{vmatrix} a & s \\ c & t \end{vmatrix} = at - cs$

We can use this new notation to discuss the solutions of a system of two equations with two variables.

Theorem (Cramer's Rule)
Consider a system of two equations with two variables

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

Let

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}$, $D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$

If $D \neq 0$, then the system has exactly one solution $\left(\dfrac{D_x}{D}, \dfrac{D_y}{D} \right)$

If $D = 0$ and $D_x = 0$ and $D_y = 0$, then the system has infinitely many solutions

If $D = 0$ and either $D_x \neq 0$ or $D_y \neq 0$, then the system has no solutions

Example:

Use Cramer's Rule to solve the system

$$\begin{cases} 3x - 6y = 24 \\ 5x + 4y = 12 \end{cases}$$

1) Compute the determinant, D, of the system

$$D = \begin{vmatrix} 3 & -6 \\ 5 & 4 \end{vmatrix} = 3 \cdot 4 - 5 \cdot (-6) = 42$$

Since $D \neq 0$, the system has a solution.

2) Compute D_x and D_y

$$D_x = \begin{vmatrix} 24 & -6 \\ 12 & 4 \end{vmatrix} = 24 \cdot 4 - 12 \cdot (-6) = 168$$

$$D_y = \begin{vmatrix} 3 & 24 \\ 5 & 12 \end{vmatrix} = 3 \cdot 12 - 5 \cdot 24 = -84$$

3) Write the solution:

$$(x, y) = \left(\frac{D_x}{D}, \frac{D_y}{D} \right) = \left(\frac{168}{42}, \frac{-84}{42} \right) = (4, -2)$$

Cramer's Rule is also applicable, in modified form, for larger systems of linear equations. We will discuss only systems of three linear equations with three variables.

We start with defining a 3x3 determinant.

Definition

A **3x3 determinant**, denoted $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, is defined as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + cdh + bgf - afh - bdi - cge$$

Note that the 2x2 determinants are obtained by crossing out the row and the column in which the numbers a,b, c are.
There are other ways to evaluate a 3x3 determinant, but we will use this method only.

Example :

Evaluate $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 1 \cdot (5 \cdot 9 - 6 \cdot 8) - 4 \cdot (2 \cdot 9 - 3 \cdot 8) + 7 \cdot (2 \cdot 6 - 3 \cdot 5) =$$

$$1(45-48) - 4(18-24) + 7(12-15) = (-3) - 4(-5) + 7(-1) = -3 + 24 - 21 = 0$$

Theorem (Cramer's Rule)

Consider the system

$$\begin{cases} ax + by + cz = s \\ dx + ey + fz = t \\ gx + hy + iz = u \end{cases}$$

Let

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, D_x = \begin{vmatrix} s & b & c \\ t & e & f \\ u & h & i \end{vmatrix}, D_y = \begin{vmatrix} a & s & c \\ d & t & f \\ g & u & i \end{vmatrix}, D_z = \begin{vmatrix} a & b & s \\ d & e & t \\ g & h & u \end{vmatrix}$$

If $D \neq 0$, then the system has exactly one solution $\left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D} \right)$

If $D = 0$ and $D_x = 0$, $D_y = 0$, $D_z = 0$ then the system has infinitely many solutions

If $D = 0$ and either $D_x \neq 0$, $D_y \neq 0$, or $D_z \neq 0$, then the system has no solutions

Example:

Use Cramer's Rule to solve the system

$$\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$$

1) Compute the determinant of the system

$$D=\begin{vmatrix}1 & 4 & -3\\ 3 & -1 & 3\\ 1 & 1 & 6\end{vmatrix}=1\cdot\begin{vmatrix}-1 & 3\\ 1 & 6\end{vmatrix}-4\cdot\begin{vmatrix}3 & 3\\ 1 & 6\end{vmatrix}+(-3)\cdot\begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix}=1\cdot((-1)\cdot 6-1\cdot 3)-4\cdot(3\cdot 6-1\cdot 3)+(-3)\cdot(3\cdot 1-1\cdot(-1))=$$

$$1(-6-3)-4(18-3)-3(3+1)=-9-60-12=-81$$

Since $D\neq 0$, system has exactly one solution

2) Compute $D_{x,}D_y, D_z$

$$D_x=\begin{vmatrix}-8 & 4 & -3\\ 12 & -1 & 3\\ 1 & 1 & 6\end{vmatrix}=(-8)\cdot\begin{vmatrix}-1 & 3\\ 1 & 6\end{vmatrix}-4\cdot\begin{vmatrix}12 & 3\\ 1 & 6\end{vmatrix}+(-3)\cdot\begin{vmatrix}12 & -1\\ 1 & 1\end{vmatrix}=-243$$

$$D_y=\begin{vmatrix}1 & -8 & -3\\ 3 & 12 & 3\\ 1 & 1 & 6\end{vmatrix}=1\cdot\begin{vmatrix}12 & 3\\ 1 & 6\end{vmatrix}-(-8)\cdot\begin{vmatrix}3 & 3\\ 1 & 6\end{vmatrix}+(-3)\cdot\begin{vmatrix}3 & 12\\ 1 & 1\end{vmatrix}=216$$

$$D_z=\begin{vmatrix}1 & 4 & -8\\ 3 & -1 & 12\\ 1 & 1 & 1\end{vmatrix}=1\cdot\begin{vmatrix}-1 & 12\\ 1 & 1\end{vmatrix}-4\cdot\begin{vmatrix}3 & 12\\ 1 & 1\end{vmatrix}+(-8)\cdot\begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix}=-9$$

3) Write the solution

$$(x,y,z)=\left(\frac{D_x}{D},\frac{D_y}{D},\frac{D_z}{D}\right)=\left(\frac{-243}{-81},\frac{216}{-81},\frac{-9}{-81}\right)=(3,-\frac{8}{3},\frac{1}{9})$$

Example:

Solve the system

$$\begin{cases}x-y+5z=-2\\ 2x+y+4z=2\\ 2x+4y-2z=8\end{cases}$$

1) Compute the determinant, D, of the system.

$$D=\begin{vmatrix}1 & -1 & 5\\ 2 & 1 & 4\\ 2 & 4 & -2\end{vmatrix}=1\cdot\begin{vmatrix}1 & 4\\ 4 & -2\end{vmatrix}-(-1)\cdot\begin{vmatrix}2 & 4\\ 2 & -2\end{vmatrix}+5\cdot\begin{vmatrix}2 & 1\\ 2 & 4\end{vmatrix}=0$$

Since D = 0, Cramer’s Rule cannot be applied.

Computing $D_{x,}D_y, D_z$ shows that $D_{x,}=D_y=D_z=0$, therefore system has infinitely many solutions. We need to use either Elimination or Substitution method to find the solution set. This has been done earlier when discussing the elimination method.

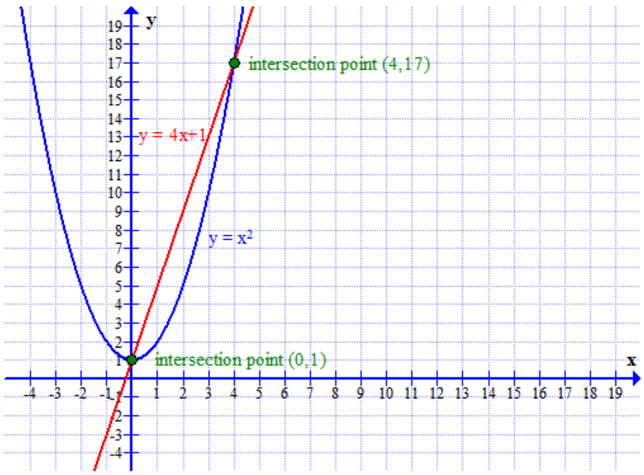
8.4 System of nonlinear equations in two variables

An **equation** is **nonlinear** if it can’t be rewritten in the form ax + by + c = 0.
Example: $x^2 + 5x + y + 4 = 0$

A system of nonlinear equations is a system that consists of one or more nonlinear equations.
A system of nonlinear equations can be solved using the Elimination or Substitution method.

Example: Solve $\begin{cases} y = x^2 + 1 \\ y = 4x + 1 \end{cases}$

Graphical method: Graph both equations and find the intersection points.



solutions: (0,1), (4,17)

Substitution method: Solve one of the equations for one of the variables, substitute into the other equation and solve it.

1) First equation is already solved for y

2) Substitute $y = x^2 + 1$ into the second equation

$$x^2 + 1 = 4x + 1$$

3) Solve for x $\quad x^2 - 4x = 0$

$$x(x-4) = 0$$

$$x = 0 \text{ or } x = 4$$

Back substitute these values to compute y

x = 0: $y = 0^2 + 1 = 1$ solution (0,1)

x = 4: $y = 4^2 + 1 = 17$ solution (4,17)

Elimination method: can be applied only when one of the variables can be eliminated. For some systems it might not be possible.

Example: Solve the system

$$\begin{cases} 5xy + 13y^2 + 36 = 0 \\ xy + 7y^2 = 6 \end{cases}$$

Multiply the second equation by (-5) and add the equations side by side

$$\begin{array}{l} 5xy + 13y^2 + 36 = 0 \\ \underline{(-5)xy + (-5)\cdot 7y^2 = (-5)\cdot 6} \\ 0 \qquad -22y^2 + 36 = -30 \end{array}$$

x variable was elimanted. The equation simplifies to

$$-22y^2 = -66$$

or

$$y^2 = 3$$

$$y = \pm\sqrt{3}$$

Substituting these values of y into the second equations, we get

If $y = \sqrt{3}$, $\sqrt{3}x + 7 \cdot 3 = 6$ and $x = -\dfrac{15}{\sqrt{3}} = -5\sqrt{3}$

and if $y = -\sqrt{3}$, $-\sqrt{3}x + 7 \cdot 3 = 6$ and $x = \dfrac{15}{\sqrt{3}} = 5\sqrt{3}$

Therefore, the solutions are $(-5\sqrt{3}, \sqrt{3}), (5\sqrt{3}, -\sqrt{3})$