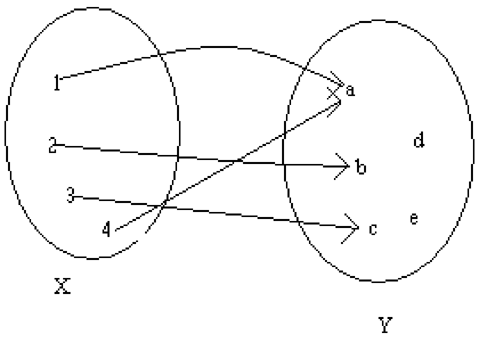
Chapter 2:

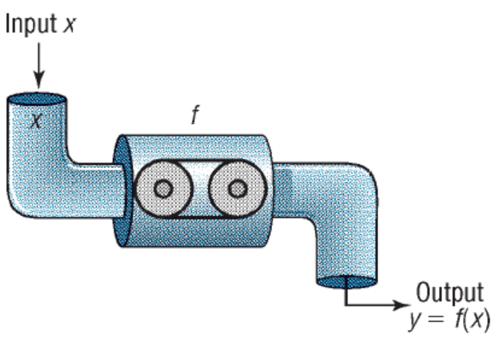
2.1 Functions: definition, notation

A **function** is a rule (correspondence) that assigns to each element x of one set , say X, one and only one element y of another set, Y.



The set X is called the **domain** of the function and the set of all elements of the set Y that are associated with some element of the set X is called the **range** of the function.
For a function described by above diagram, the domain is the set X= {1,2,3,4} and the range is the set {a,b,c}.

The elements of a set X are often called **inputs** and the elements of the set Y are called **outputs** and a function can be visualized as a machine, that takes x as an input and returns y as an output.



The **domain of a function** is the set of all inputs x that return an output.
The **range of a function** is the set of all outputs y.

We will deal with functions for which both domain and the range are the set (or subset) of real numbers

Most often functions in mathematics are defined by a formula or by a graph.
Very often we define a function by giving a **formula or an equation** which specifies how the *y* is obtained when *x* is given.

Example: The equation $y = 3x + 2$ defines a function since we can easily find the output y when the input, x, is given. To obtain the y that correspond to a given x, multiply x by 3 and add 2 to the result. For example, if x is 1 then $y = 3(1) + 2 = 5$. So, every x has a corresponding y value.

We say that a function is **given explicitly** by an equation, if the equation is of the form y = (formula that contains only x variable). We say that a function is given **implicitly**, if it is given by an equation that is not in the above form. For example, y = 3x + 2 is a function given in an explicit form, and 3x - y + 2 = 0 is the same function given in an implicit form.

When we use a formula to define a function we often give a name to a function (f,g,h,etc.) and use a special notation for the output, y. If x is the input then we denote the output by f(x) (read "f of x").

Caution: f(x) is not a multiplication of f and x. It is an entity that can't be split. f(x) denotes output that corresponds to the input x.

For example, instead of writing y = 3x+2, we often write f(x) = 3x+2. With this notation f(1) denotes the output (y) that corresponds to the input 1, that is f(1) = 3·(1)+2= 5.
If we say that f(4) = -5, then this means that when the input, x, is 4, the output, y , is -5.

An equation defines a function if it can be solved for y and the solution is unique.

Example: 3x+2y = 4 is a function, because we can solve it for y, y = (-3/2)x + 2 and the solution is unique. The equation $x^2 + y^2 = 1$ is not a function, because when we solve it for y, we get $y = \pm\sqrt{1-x^2}$, two solutions for $-1 \leq x \leq 1$. For example when x = 0, then there are two values of y (± 1) that correspond to x = 0. Hence, this equations does not describe a function.

Example: Determine whether the equation $2x + 4y^2 - 5 = 0$ defines y as a function of x.

Solution : Let's solve the equation for y:

$$4y^2 = 5 - 2x$$

$$y^2 = \frac{1}{4}(5-2x)$$

$$y = \pm\sqrt{\frac{1}{4}(5-2x)}$$

Since there are two solutions, this equation does not define a function

The **graph of a function** is the set of all points (x, f(x)), where x belongs to the domain of f.
A point (a,b) is on the graph of a function *f*, if and only if *b = f(a).*

If f is a function and a is in its domain, then there is one and only one value b that corresponds to a. Therefore, there is only one point on the graph of f that has a as x-coordinate. This leads us to the

Vertical Line Test

Theorem:

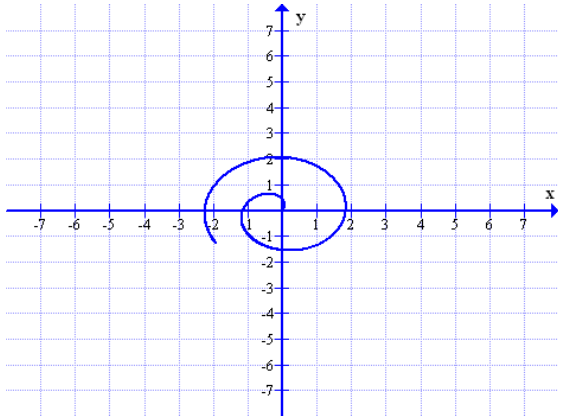
If every vertical line crosses the graph at most at one point, then the graph represents a function.

The domain of a function given by a graph is the set of all x -coordinates of the points on the graph. **The range** is the set of all y-coordinates of the points on the graph.

Example:

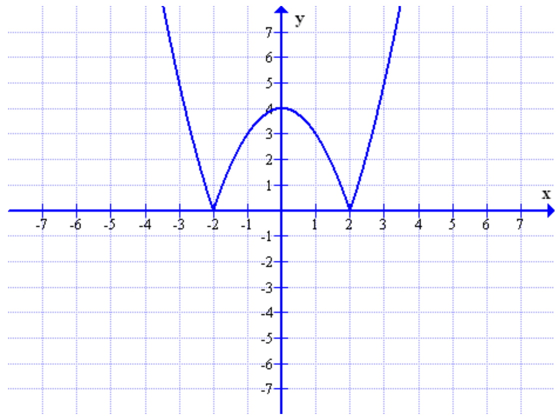
Which of the following is the graph of a function?

a)



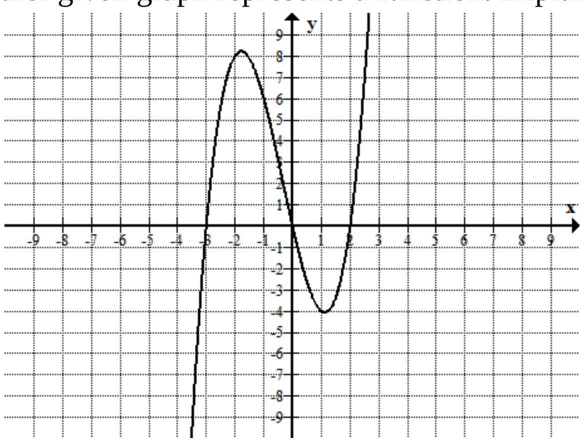
The graph fails the Vertical Line test, so the graph does not represent a function

b)

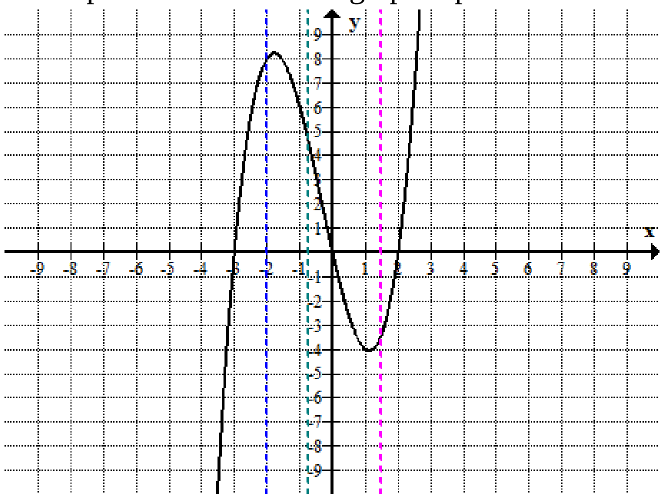


The graph passes the Vertical Line test so the graph represents a function

Example: Determine whether given graph represents a function. Explain why or why not.



Solution: We perform the Vertical Line Test, and notice that every vertical line crosses the graph at most at one point. Therefore the graph represents a function.



When we use a formula to define a function we often give a name to a function (f,g,h,etc.) and use a special notation for the output, y. If f is a function and x is the input then we denote the corresponding output by f(x) (read "f of x").

Caution: f(x) is not a multiplication of f and x. It is an entity that can't be split. f(x) denotes output that corresponds to the input x.

Example: Let $f(x) = 3x^2 - 2x - 4$

Find f(0), f(1), f(-2), f(-x), -f(x), f(x+1), f(2x), f(x+h)

$f(0) = 3(0)^2 - 2(0) - 4 = -4$

$f(1) = 3(1)^2 - 2(1) - 4 = -3$

$f(-2) = 3(-2)^2 - 2(-2) - 4 = 12$

$f(-x) = 3(-x)^2 - 2(-x) - 4 = 3x^2 + 2x - 4$

$f(x+1) = 3(x+1)^2 - 2(x+1) - 4 = 3(x^2 + 2x + 1) - 2x - 2 - 4 = 3x^2 + 6x + 3 - 2x - 6 = 3x^2 + 4x - 3$

$f(2x) = 3(2x)^2 - 2(2x) - 4 = 3(4x^2) - 4x - 4 = 12x^2 - 4x - 4$

$f(x+h) = 3(x+h)^2 - 2(x+h) - 4 = 3(x^2 + 2xh + h^2) - 2x - 2h - 4 = 3x^2 + 6xh + 3h^2 - 2x - 2h - 4$

$-f(x) = -(3x^2 - 2x - 4) = -3x^2 + 2x + 4$

Example: For the function $g(x) = \frac{3x-1}{2-x}$, find g(-3), g(0), g(2), g(-x), g(x-2), g(3x), g(x+h).

Solution: $g(-3) = \frac{3(-3)-1}{2-(-3)} = \frac{-9-1}{2+3} = \frac{-10}{5} = -2$

$$g(0) = \frac{3(0)-1}{2-(0)} = \frac{0-1}{2-0} = \frac{-1}{2}$$

$$g(2) = \frac{3(2)-1}{2-(2)} = \frac{6-1}{2-2} = \frac{5}{0}$$ this value is not defined; 2 does not belong to the domain of g

$$g(-x) = \frac{3(-x)-1}{2-(-x)} = \frac{-3x-1}{2+x}$$

$$g(x-2) = \frac{3(x-2)-1}{2-(x-2)} = \frac{3x-6-1}{2-x+2} = \frac{3x-7}{4-x}$$

$$g(3x) = \frac{3(3x)-1}{2-(3x)} = \frac{9x-1}{2-3x}$$

$$g(x+h) = \frac{3(x+h)-1}{2-(x+h)} = \frac{3x+3h-1}{2-x-3}$$

Simplifying the difference quotient $\frac{f(x+h)-f(x)}{h}$

Example :

Find $\frac{f(x+h)-f(x)}{h}$ (**the difference quotient** for f) for f(x) = 3x^2 -2x - 4

$$\frac{f(x+h)-f(x)}{h}=\frac{\left(3(x+h)^2-2(x+h)-4\right)-(3x^2-2x-4)}{h}=\frac{3(x^2+2xh+h^2)-2x-2h-4-3x^2+2x+4}{h}=$$

$$\frac{3x^2+6xh+3h^2-2h-3x^2}{h}=\frac{6xh+3h^2-2h}{h}=\frac{h(6x+3h-2)}{h}=6x+3h-2$$

Example :

Find $\frac{g(x+h)-g(x)}{h}$ (**the difference quotient** for g) for $g(x)=\frac{3}{2-x}$

Solution:

$$\frac{g(x+h)-g(x)}{h}=\frac{\frac{3}{2-(x+h)}-\frac{3}{2-x}}{h}=\frac{\frac{3}{2-x-h}-\frac{3}{2-x}}{h}=\frac{1}{h}\left[\frac{3(2-x)-3(2-x-h)}{(2-x-h)(2-x)}\right]=$$

$$\frac{1}{h}\left[\frac{6-3x-6+3x+3h}{(2-x-h)(2-x)}\right]=\frac{1}{h}\cdot\frac{3h}{(2-x-h)(2-x)}=\frac{3}{(2-x-h)(2-x)}$$

Domains of functions

When a function f is given by a formula and its domain is not given, then it is assumed that the **domain is the largest set of real numbers for which f (x) can be computed and is a real number. Often it is more practical to ask what values of x cannot be used in a formula.**

At this point, to find a domain of a function you should examine the formula for the function and ask two questions

A) Does the formula have a **square root** (or even index radical)?
IF yes, we have to make sure that the expression inside the radical (called radicand) is not negative. For example, if $f(x)=\sqrt{3x-1}$ then we cannot allow 3x-1 to be negative, as the square root of a negative number does not exist.

and

B) Does the formula contain a **variable in the denominator**?
If yes, we have to guarantee that the denominator does not become 0. For example, if $f(x)=\frac{2x^2+3}{x^2+2x-1}$, then we can allow any value for x as long x^2 +2x -1 is **not** zero since the division by zero is not defined.

If the formula contains a variable in the denominator, the domain consists of all real numbers **except** the ones that make the denominator zero.

To find the domain in this case:

(i) Solve the equation: denominator = 0

(ii) Write the domain: Df ={x|x ≠ (list the solutions found in (i))}

Example : Find the domain of $f(x) = \dfrac{3x+2}{x^3 - 5x}$

Since there is a variable in the denominator, we must guarantee that the denominator is not zero.

(i) Solve the equation: denominator = 0

$x^3 - 5x = 0$

$x(x^2 - 5) = 0$

$x = 0$ or $x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5}$

(ii) Write the domain: $Df = \{x \mid x \neq 0, \sqrt{5}, -\sqrt{5}\}$

Example: Find the domain of $f(x) = \dfrac{2x^2+3}{x^2+2x-1}$

Solution: Since there is a variable in the denominator, we must guarantee that the denominator is not zero.

(i) Solve the equation: denominator = 0

$x^2 + 2x - 1 = 0$

This is a quadratic equation. Since it cannot be factored, we must use the quadratic formula to solve it:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2 \cdot 1} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

(ii) Write the domain: $Df = \{x \mid x \neq -1 + \sqrt{2}, -1 - \sqrt{2}\}$

If the formula contains a **square root** or even index radical then the domain consists of all values x for which the radicand is nonnegative

To find the domain in this case:

(i) Solve the inequality: radicand ≥ 0

(ii) Write the domain: Df = solution from (i) in the interval notation

Example: Find the domain of $f(x) = \sqrt{8 - 2x}$

(i)

$8 - 2x \geq 0$

$-2x \geq -8$

$x \leq \dfrac{-8}{-2}$

$x \leq 4$

(ii) $Df = (-\infty, 4]$

Example: Find the domain of $f(x) = \sqrt{3x - 1}$

Solution: Since the formula contains the square root, we have to guarantee that the expression inside the radical is nonnegative

(i) We must have : $3x - 1 \geq 0$

This means that $3x \geq 1$

Dividing both sides by 3 we get : $x \geq 1/3$

(ii) Write the domain $Df = [1/3, +\infty)$

Sometimes the formula for a function might contain both the even index radical and a variable in the denominator or might not contain any of these .

Example: Find the domain of $f(x) = \dfrac{2x^2 - 2}{\sqrt{x+1} - 3}$

Solution: Here we have a variable in the denominator and even index radical. Therefore, we should make sure that the denominator is not zero: $\sqrt{x+1} - 3 \neq 0$, and that the radical is well defined: $x + 1 \geq 0$.

First we solve $\sqrt{x+1} - 3 = 0$

$\sqrt{x+1} = 3$

$x + 1 = 3^2$ (after squaring both sides)

$x = 8$

Since we want $\sqrt{x+1} - 3 \neq 0$, x cannot be 8, $x \neq 8$

Second, we solve the inequality $x + 1 \geq 0$

$x \geq -1$

Therefore, f(x) will be well defined if $x \geq -1$ and $x \neq 8$. This means that $Df = [-1, 8) \cup (8, +\infty)$.

2.2 The graph of a function- general properties

The **graph of a function f** is the set of all points (x,y), where x belongs to the domain of f and y = f(x). If a is in the domain of f then the point (a, f(a)) is on the graph of f.

Example: Let f(x) = $x^2 - 2$.

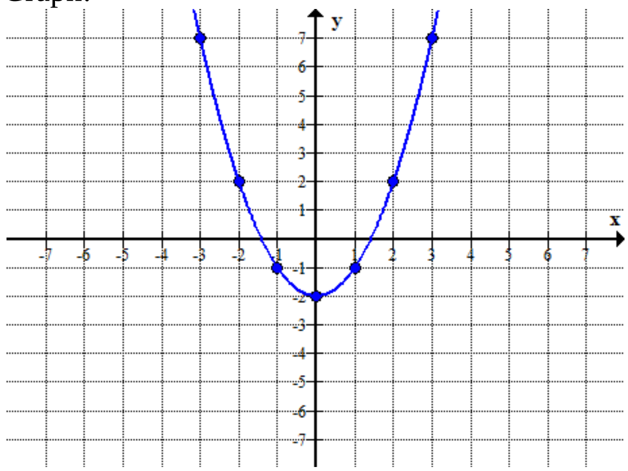
a) If x = 1, what point is on the graph of f?
   If x = 1 then y = f(1) = 1^2 -2 = -1. Therefore , the point (1,-1) is on the graph of f.
b) Is the point (2,-1) on the graph of f?
   Point (2,-1) would be on the graph of f is -1 = f(2). But f(2) = 2^2 -2 = 2. Thus, the point (2,-1) is not on the graph.
c) Sketch the graph of f
   To sketch the graph of f we need to plot the points (x,f(x)), where x is in the domain of f.
   We can select some values for x, find corresponding y, plot the points (x,y) and join them with a continuous curve in the order of increasing x values

   Table:

x	y= f(x) = x^2-2	(x,y)
-3	$(-3)^2$ -2 = 7	(-3,7)
-2	4-2= 2	(-2,2)
-1	1-2= -1	(-1,-1)
0	0-2= -2	(0,-2)
1	1-2= -1	(1,-1)

2	4-2 = 2	(2,2)
3	9-2=7	(3,7)

Graph:



A graph of an equation is the graph of a function if it passes the **Vertical Line Test**: *If every vertical line crosses the graph at most at one point, then the graph represents a function.*

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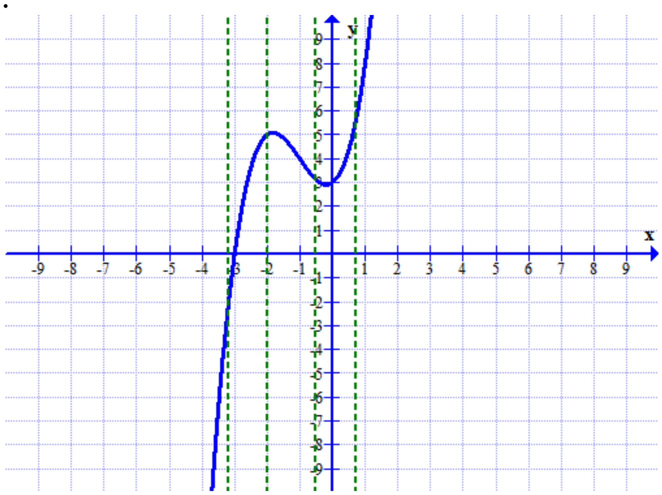
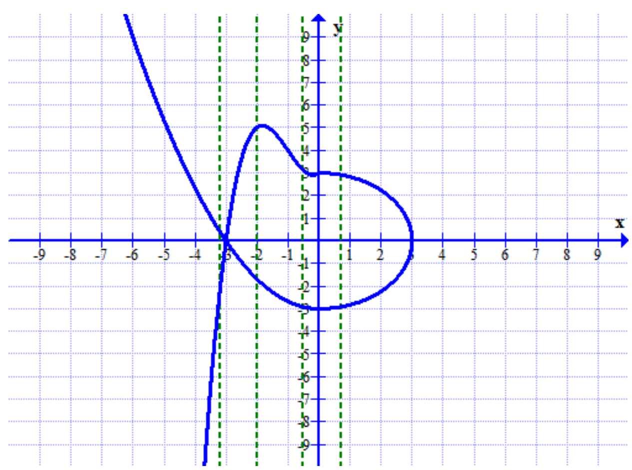


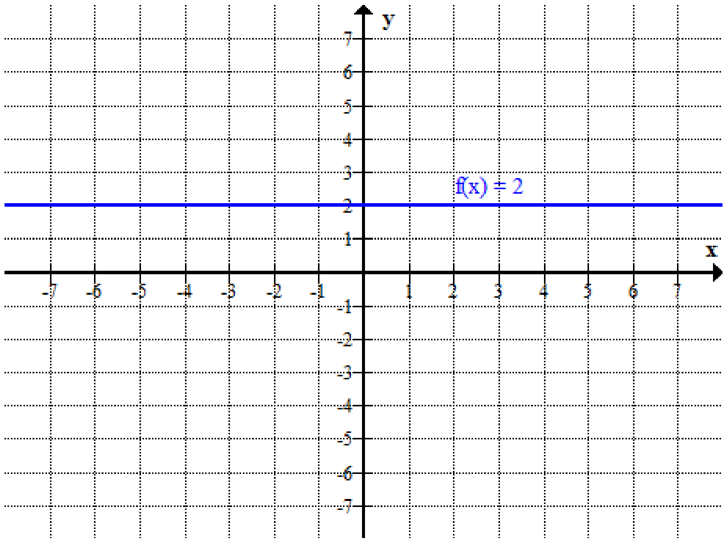
A function



not a function

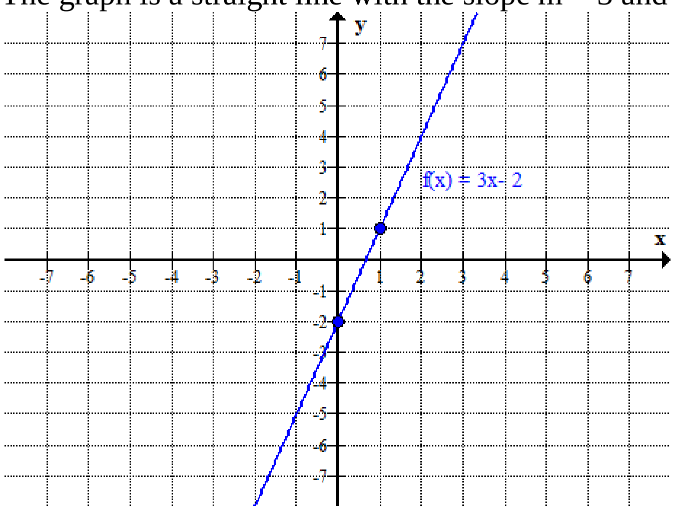
Graphs of some functions:

Constant function: f(x) = C, where C is a constant
Example: Graph f(x) = 2



Linear function : $f(x) = ax+b$, where a and b are real numbers.
Its graph is a straight line $y = ax + b$

Example: Graph $f(x) = 3x - 2$
The graph is a straight line with the slope $m = 3$ and y-intercept (0,-2)

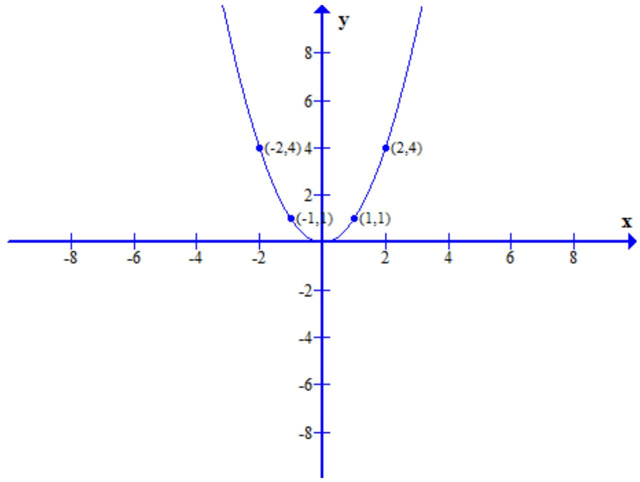


Square function: $f(x) = x^2$

The graph of this function is the graph of the equation $y = x^2$. We construct the table, plot the points and connect them with a continuous curve.

x	$y = x^2$
-2	4
-1	1
0	0

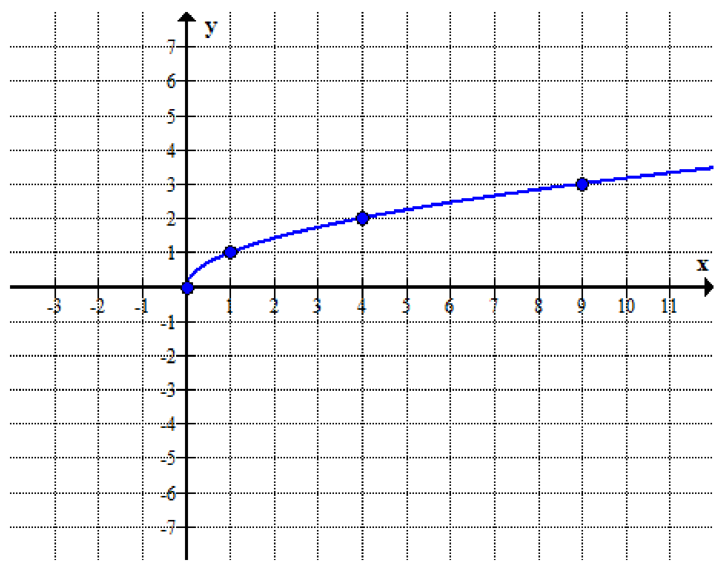
1	1
2	4



Square root function f(x) = $\sqrt{x}$. It's domain is the interval [0, +∞)

The graph of this function is the graph of the equation y = $\sqrt{x}$**.** We construct the table, plot the points and connect them with a continuous curve.

x	y = $\sqrt{x}$
0	0
1	1
4	2
9	3
16	4



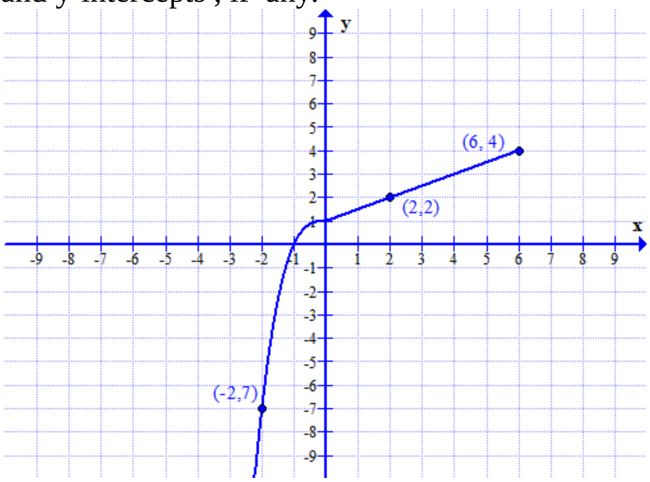
Reading the graph of a function: domain, range, intercepts

If the graph of a function is given then we can determine its domain, range, x- and y- intercepts as well as find values f(x) for a specific value of x.
The **domain** of a function given by a graph is the set of all x, such that for some y, the point (x, y), is on the graph. Or, in other words, if the vertical line passing through x crosses the graph at some point.
The **range** of a function is the set of all y, such that, for some value x, the point (x,y) is on the graph. Or, in other words, if the horizontal line passing through y crosses the graph at some point.

To find value f(a) from the graph, locate the point on the graph whose x-coordinate is a. The y-coordinate of that point is f(a).

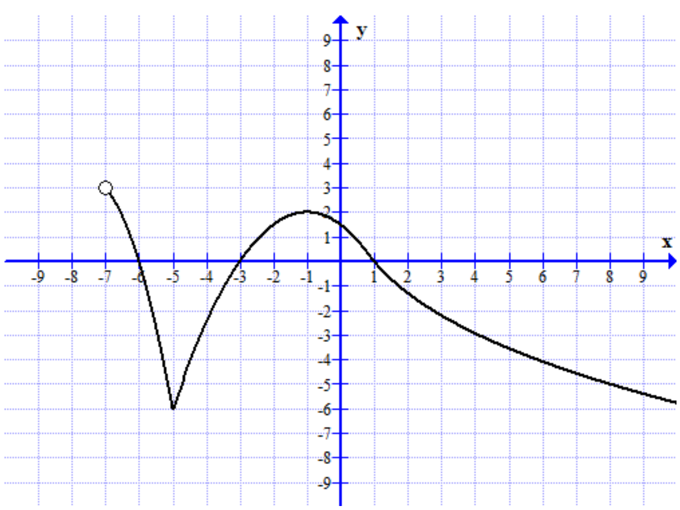
The **x-intercepts** are the points where the graph crosses or touches the x- axis. They are found by solving the equation f(x) = 0. They are the points at which f has value zero, and therefore they are often called the **zeros of function** f.
The **y-intercept** is a point where the graph crosses or touches the y- axis. To find y-intercept, compute f(0). A function can have at most one y-intercept.

Example: The graph of a function f is given below. Find the domain and the range, f(-2), f(4) and the x- and y-intercepts , if any.



If you follow the graph from left to right, the x –coordinate changes from - ∞ to 6, so
Df = (-∞, 6]. At the same time, the y-coordinate changes from - ∞ to 4, so Range of f= (-∞, 4].
To find f(- 2), find the point on the graph with x-coordinate -2. This point is (-2,-7). The y-coordinate of this point is -7, hence f(-2) = -7. Similarly, we can find that f(4) = 3.
The graph crosses x-axis at (-1,0), so the x-intercept is (-1,0). The y-intercept is (0,1)

Example: Explain why the graph below represents a function. Find its domain and range. Find values of this function when x = -3, -1, 8. List the intercepts.

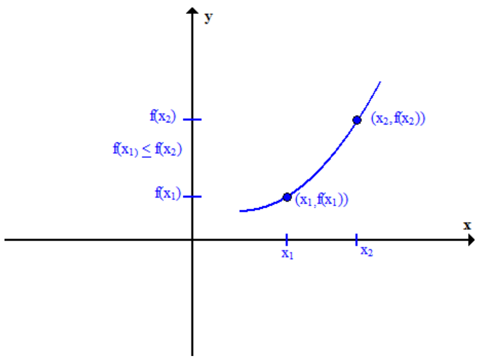


Solution: The graph represents a function because it passes the Vertical Line Test. Any vertical line crosses the graph at most at one point.
Domain = (-7, +∞)

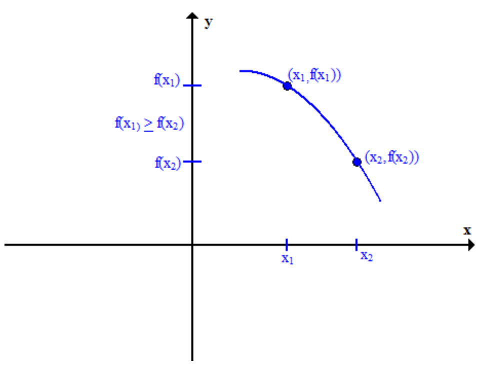
Range = $(-\infty, 3)$
f(-3) = 0
f(-1)= 2
f(8) = -5
x- intercepts are: (-6,0), (-3, 0), (1,0)
y-intercept is: (0,3/2)

Reading the graph of a function: increasing, decreasing, constant functions

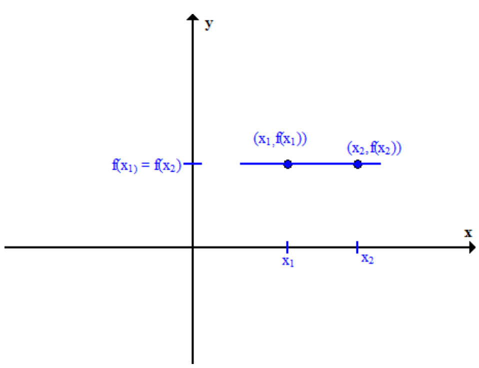
A function is **increasing** on an interval (a, b), if for every two values x_1 and x_2 in that interval, $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ (y increases when x increases). A function is increasing on an interval if the graph is rising on this interval.



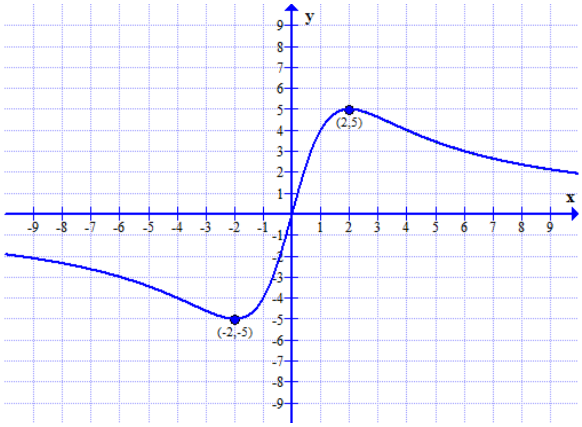
A function is **decreasing** on an interval (a, b), if for every two values x_1 and x_2 in that interval, $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ (y decreases when x increases). A function is decreasing on an interval, if the graph is falling on this interval.



A function is **constant** on an interval (a, b), if for every two values x_1 and x_2 in that interval, $f(x_1) = f(x_2)$ (y remains the same). A function is constant on an interval, if the graph is horizontal on this interval.

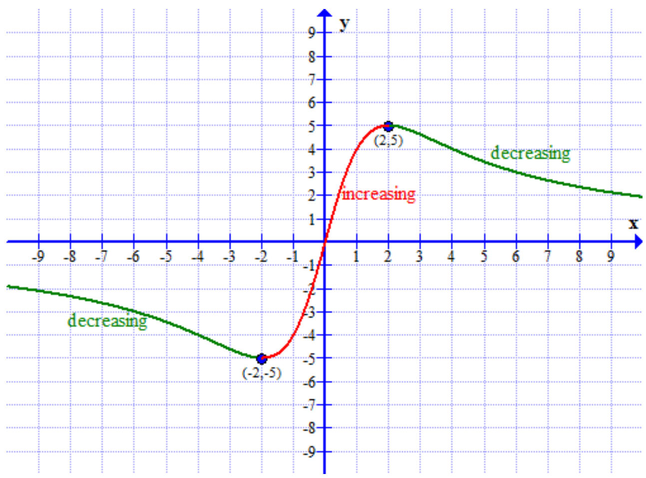


Example: Find the intervals on which the following function is increasing/ decreasing/constant



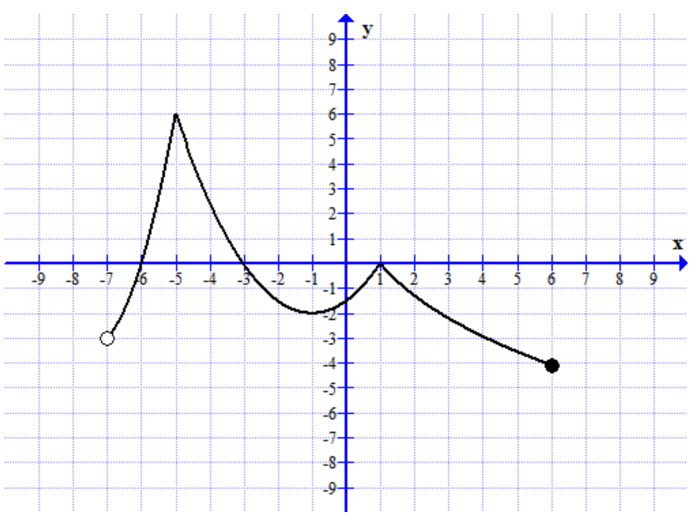
(i) Identify the parts that are increasing/ decreasing/ constant

(ii)



(iii) Determine how the x-coordinate changes when you follow each part, and write it in the interval notation.
In the example above, graph is rising (function is increasing) when x- changes from -2 to 2, so f is increasing on (-2,2).

Example: Find the intervals on which the following function is increasing/ decreasing/constant

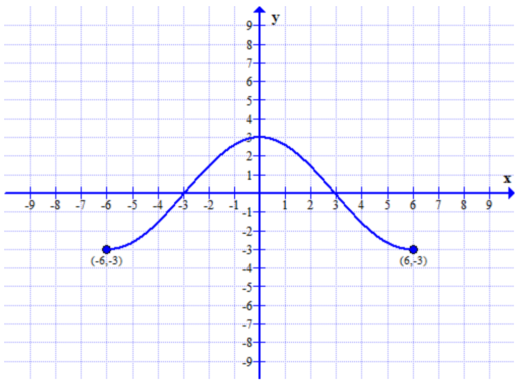


Solution: We can see that the graph is rising when x changes from -7 to -1 (on the interval (-7,-5)) ; then it is falling when x changes from -5 to -1 (on the interval (-5,-1)); then it rises when x changes from -1 to 1 (on the interval (-1,1)) and finally the graph is falling when x changes from 1 to 6 (on the interval (1,6])

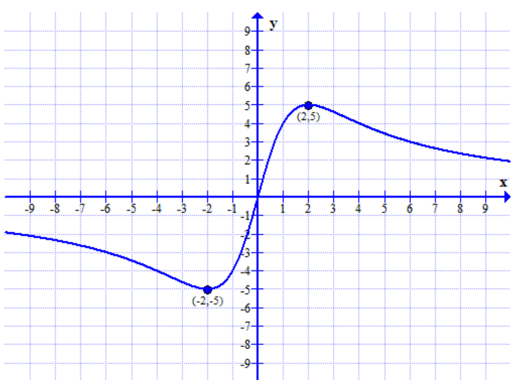
Function is increasing on the intervals (-7, -5), (-1, 1) and decreasing on (-5, -1), (1, 6]

Reading the graph of a function: even, odd functions

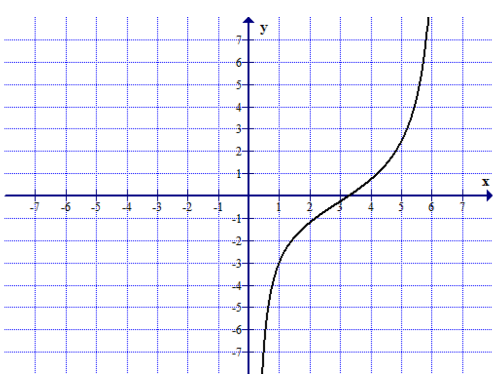
A function f is **even**, if for every x in its domain, -x is also in the domain and f(-x) = f(x). This means that for an even function, if (x,y) is on the graph than (-x,y) is also on the graph. The graph of an even function is **symmetric with respect to the y-axis.** The graph below shows an even function.



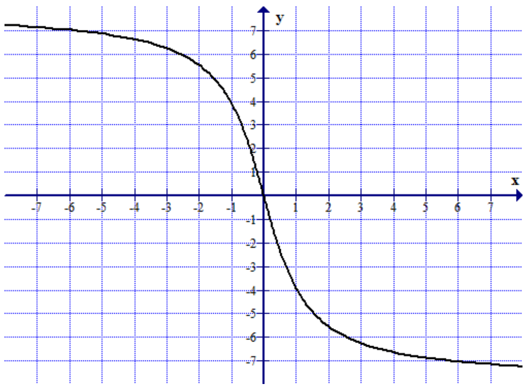
A function f is **odd**, if for every x in its domain, -x is also in its domain and f(-x) = -f(x). This means that for an odd function, if (x,y) is on the graph than (-x,-y) is also on the graph. The graph of an odd function is **symmetric with respect to the origin.** The graph below shows an odd function



A function might be neither odd nor even. The graph below is neither symmetric about the y-axis or the origin.

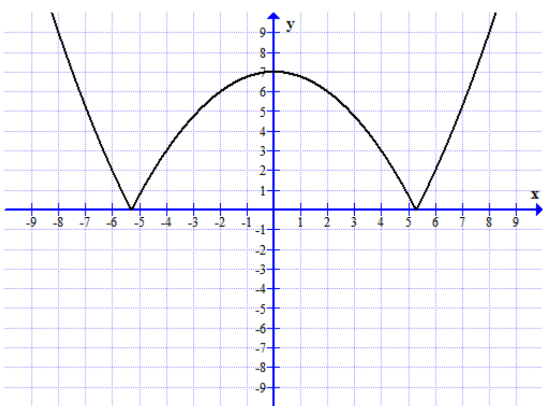


Example: Is the function below even, odd or neither. Explain



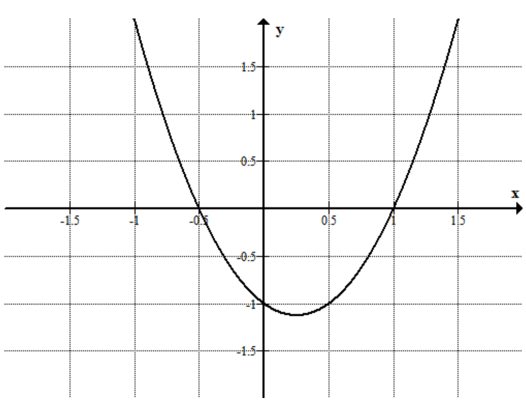
The graph is not symmetric with respect to the y-axis, so this function is not even. The graph, however, is symmetric about the origin, so it is an odd function. Symmetry about the origin can be noticed by considering a line through the origin and a point on the graph (a,b). The line will cross the graph again, at a point that has coordinates (-a, -b). Check, for example, points (1,-4) and (-1,4)

Example: Is the function below even, odd or neither. Explain



Solution: The graph is symmetric about the y-axis, so it is an even function.

Example: Is the function below even, odd or neither. Explain.



Solution: This function is neither odd nor even. The graph is not symmetric about the origin nor the y-axis. Consider for example the points (1,0) and (-.5, 0). They are not symmetric about the y-axis nor about the origin

Using the formula of a function to determine whether a function is odd or even

Knowing whether a function is even or odd helps determine properties of the graph as well as helps in graphing functions.

We use the definitions to check whether function is odd, even or neither. Here is the procedure

(i) Evaluate f(-x) and simplify
(ii) Compare f(-x) with f(x). If the formulas are the same then f is even. If the formulas are not the same, then f is not even.
(iii) Write the formula for –f(x)
(iv) Compare f(-x) with –f(x). If the formulas are the same then f is odd. If the formulas are not the same then f is not odd.

Remarks: i) If f is even, then it cannot be odd and vice versa

ii) There are functions that are neither odd nor even

Example: Check whether $f(x)=\frac{3x}{x^2-1}$ is even, odd or neither

(i) $f(-x)=\frac{3(-x)}{(-x)^2-1}=\frac{-3x}{x^2-1}$
(ii) f(x) and f(-x) are not the same, so f is not even
(iii) $-f(x)=-\frac{3x}{x^2-1}=\frac{-3x}{x^2-1}$
(iv) f(-x) and –f(x) are the same, so f is odd. Hence, the graph of f(x) is symmetric with respect to the origin.

Example: Check whether $f(x)=3x^2+2x-5$ is even, odd or neither

Solution: Let's start with computing and simplifying f(-x).

$$f(-x)=3(-x)^2+2(-x)-5=3x^2-2x-5$$

Comparing f(x) and f(-x), we see that the formulas are not the same, therefore, function f is not even.

We check now whether f is an odd function. We compute and simplify -f(x)

$$-f(x)=-(3x^2+2x-5)=-3x^2-2x+5$$

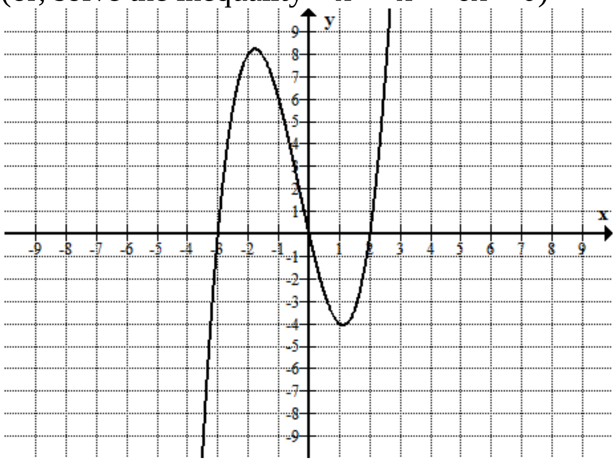
Comparing f(-x) and –f(x) shows that the formulas are not the same, which means function f is not odd.

Hence f is neither even nor odd. The graph of function f is not symmetric with respect to the y axis nor with respect to origin.

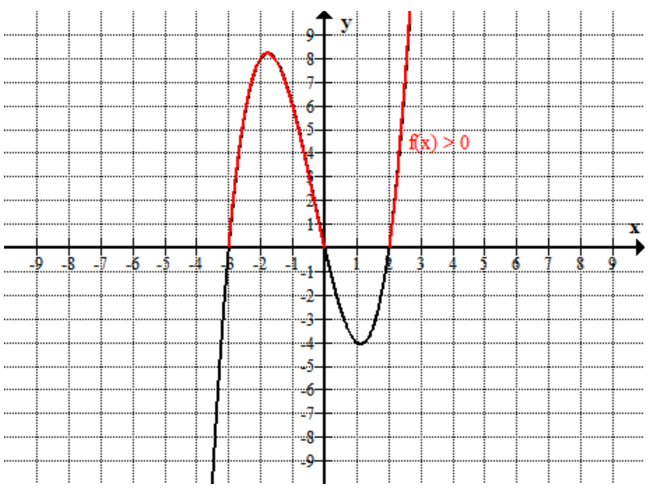
Reading the graph of a function: determining intervals on which f(x) > 0, f(x) < 0

Sometimes we can use the graph of a function to solve an inequality of the form f(x) > 0, f(x) $\geq$ 0, f(x) < 0, f(x) $\leq$ 0.
Note that if f(x) > 0 then value y= f(x) is positive, which means that the point (x, y) is above the x-axis. If f(x) < 0, then the point (x, y) will lie below the x-axis. So, to find all values of x for which f(x) >0 (or (f(x) < 0) (or to solve the inequality f(x) >0) we must determine which part(s) of the graph is(are) above (below) the x-axis and determine values of x that generate this graph.

Example: The graph of the function $f(x) = x^3 + x^2 - 6x$ is given below. Find values of x for which f(x) >0. (or, solve the inequality $x^3 + x^2 - 6x > 0$)



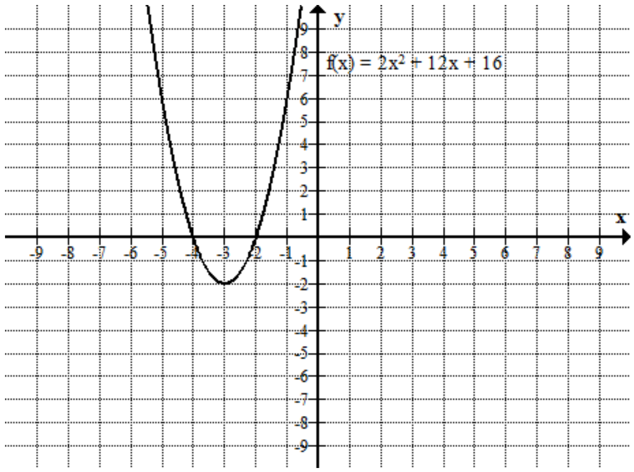
(i) identify the parts of the graph that are above the x-axis



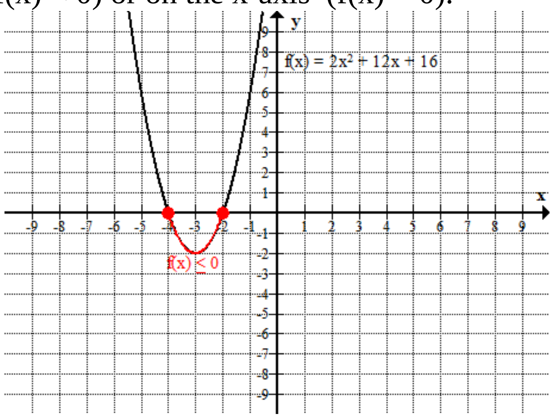
(ii) determine values of x that generate the part of the graph with a desired property.

f(x) > 0 when x is in the interval (-3, 0) or (2, +∞). Note that the endpoints are not included, because f(-3) = 0, f(0) = 0, f(2) = 0 and we want f(x) to be strictly greater than 0.

Example: The graph of the function $f(x) = 2x^2 + 12x + 16$ is given below. Solve the Inequality $2x^2 + 12x + 16 \leq 0$



Solution: Since we are looking for values of x for which $f(x) \leq 0$, we should identify the parts of the graph that are below ($f(x) < 0$) or on the x-axis ($f(x) = 0$).



Therefore, $2x^2 + 12x + 16 \leq 0$ if x is in [-4,-2]

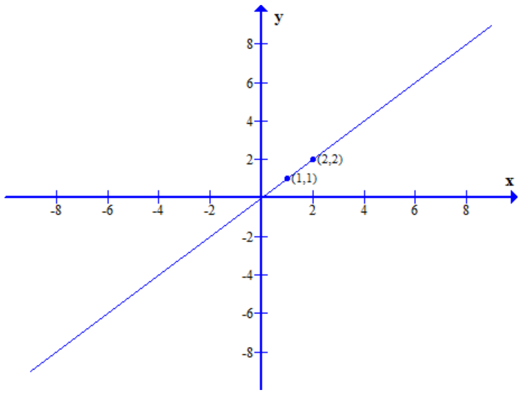
Library of functions

Here are the graphs of other functions that you must be able to recognize and draw. Recall that the graph of a function f(x) is the graph of the equation y = f(x), that is the set of all points (x,f(x)) where x changes through the domain of f. We obtain these graphs by making a table x|y , plotting the points (x,y) and joining them with a continuous curve. The more points we have, the more accurate the graph.

(A) **f(x) = x identity function**

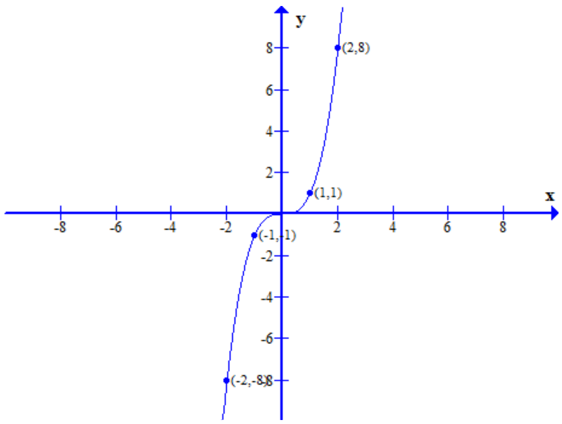
The graph of this function is the graph of an equation y = x, that is, it is the straight line with the slope

m = 1 and the y-intercept (0,0)

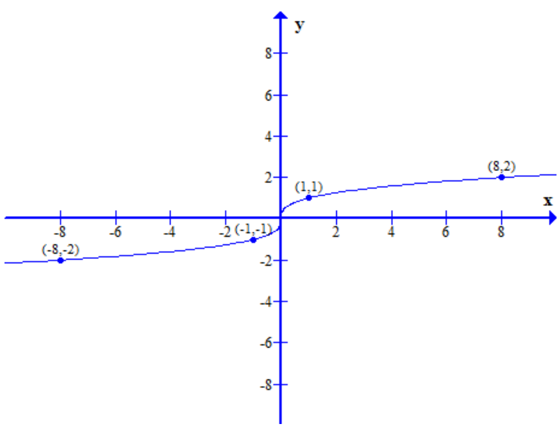


Similarly we can obtain the graphs of the functions below

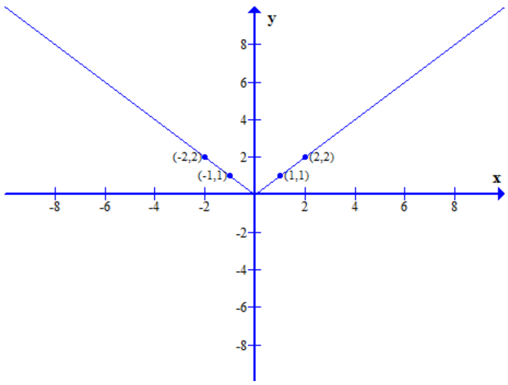
(B) **f(x) = x^3** **cube function**



(C) f**(x) =** $\sqrt[3]{x}$ **cube root function**



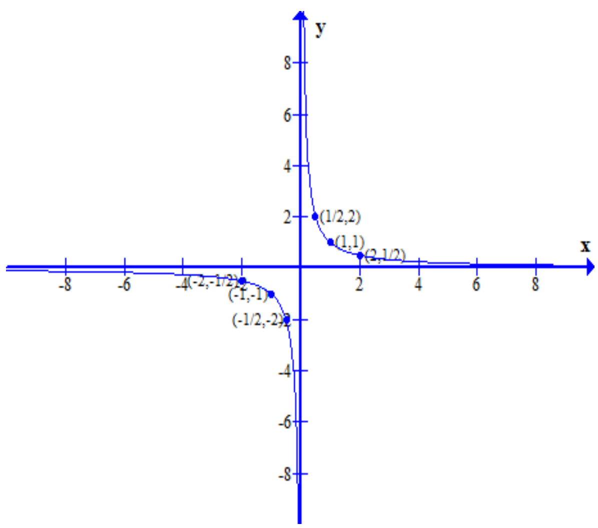
(D) **f(x) =|x| absolute value function**



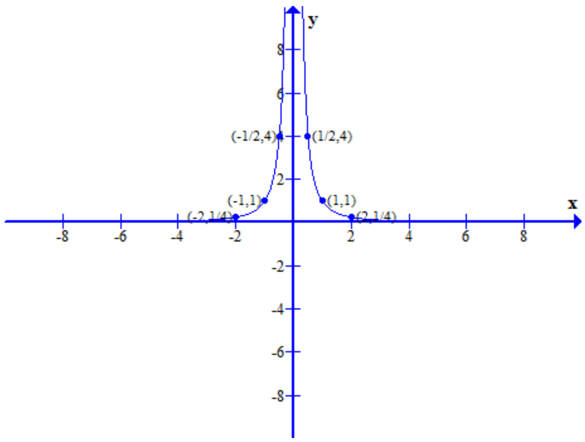
(E) **f(x) =** $\frac{1}{x}$ **reciprocal function**

Note that the domain of this function is the set of all real numbers except x = 0. Therefore, the graph will consist of two parts- one for x < 0 and one for x > 0. Notice also that if the value of |x| is large, its reciprocal will be close to 0 and if the value of x is close to 0, its reciprocal will have large absolute value, as can be seen in the table below. Therefore, the graph will come closer and closer to the x axis as x becomes larger and larger (positive or negative) and will come closer and closer to the y-axis as x approaches zero. We say that the x –axis is a **horizontal asymptote** and that the y- axis is a **vertical asymptote** for the graph of this function.

x	$y = \frac{1}{x}$		x	$y = \frac{1}{x}$
-10	- 0.1		0.1	10
-100	-0.01		0.01	100
-1000	-0.001		0.001	1000
-10, 000	-0. 0001		0. 0001	1000
-100,000	-0.00001		0.00001	100 000



(F) f(x) = $\frac{1}{x^2}$ the reciprocal of the square function



Piecewise functions

A function that is defined by more than one formula is called a **piecewise function**. Such functions are written using the brace {

Example:

$$f(x) = \begin{cases} 2x + 3 & if \quad x < 1 \\ |x| & if \quad x \geq 1 \end{cases}$$

This function is defined by formula f(x) = 2x+3 only when x < 1, and by formula f(x) = |x| only when x $\geq$ 1.

To find f(a), you must first determine whether a < 1 or a $\geq$ 1. If a < 1, then you will use the first formula, f(x) = 2x +3. If a $\geq$ 1, then you will use the second formula, f(x) = |x|. For example, to evaluate f(-2), we first notice that x = -2 < 1, so we use the formula reserved for x < 1, which is f(x) = 2x +3.

Hence, f(-2) = 2(-2)+ 3 = -1.

Example: A function f is given below. Find values f(-2), f(0), f(3/2)

$$f(x) = \begin{cases} x^2 + 3, if \ x \leq 0 \\ \frac{1}{x} \ , if \ x > 0 \end{cases}$$

To evaluate f(-2), we first notice that x = -2 $\leq$ 0, so we use the formula reserved for x $\leq$ 0, which is f(x) = x^2 +3.Therefore, f(-2) = $(-2)^2$ + 3= 4+3 = 7.

To evaluate f(0), we first notice that x = 0 $\leq$ 0, so we use the formula reserved for x $\leq$ 0, which is

$f(x) = x^2 + 3$.Therefore, $f(0) = (0)^2 + 3 = 0+3 = 3$.

To evaluate f(3/2), we first notice that x = 3/2 $\geq$ 0, so we use the formula reserved for x > 0, which is

f(x) = 1/x .Therefore, f(3/2) = 1/(3/2) = 2/3

Example : Find f(-1), f(2) for the function given below

$$f(x) = \begin{cases} 2x - 5 \text{ , } if \;\; x \leq -1 \\ \sqrt{x+2} \;\; \text{, } if \;\; x > -1 \end{cases}$$

Solution: To compute f(-1), we first notice that x = -1 $\leq$ -1, therefore we will use the expression 2x-5.

f(-1) = 2(-1) - 5 = -7

To compute f(2), we notice that x = 2 > -1, therefore we will use the expression $\sqrt{x+2}$.

$$f(2) = \sqrt{2+2} = \sqrt{4} = 2$$

Graphing a piecewise function

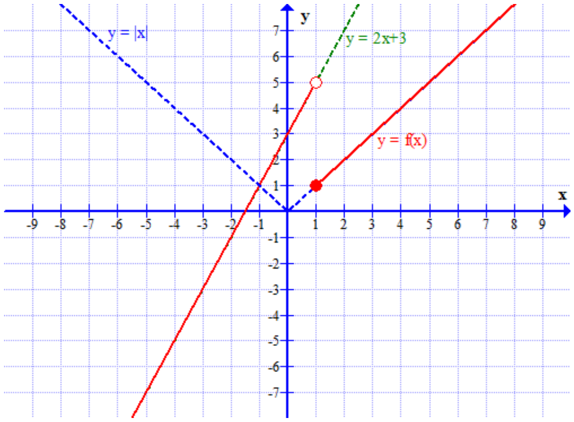
To graph a piecewise function we must graph each function involved in the formula and take the piece of that graph that corresponds to the given values of x.

Example:

$$f(x) = \begin{cases} 2x + 3 \; if \quad x < 1 \\ |x| \quad\;\; if \quad x \geq 1 \end{cases}$$

First we graph y = 2x + 3. The graph is a straight line with the y-intercept (0,3) and the slope m = 2. Since f(x) = 2x+3 only for x < 1, we consider only the part of the line that corresponds to x < 1. The point (1,5) is not a part of the graph of f(x), since f(1) = |1| = 1. Therefore we draw an open circle at this point.

Next, we draw the graph of y = |x| and take the part of this graph that corresponds to x $\geq$ 1. Since f(1) = 1, point (1,1) is a part of the graph and is drawn as a solid disc.The dashed parts of the graphs of y= 2x+3 and y = |x| should be erased.

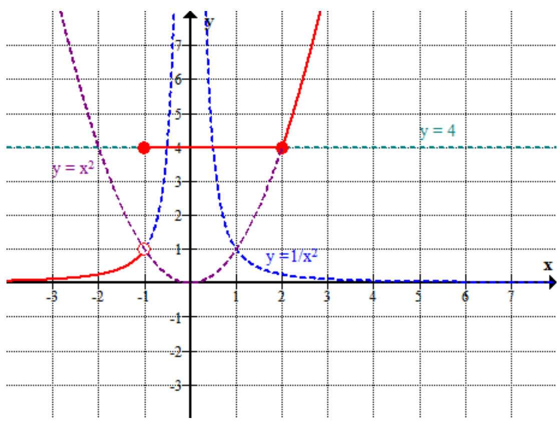


Example: Graph the function

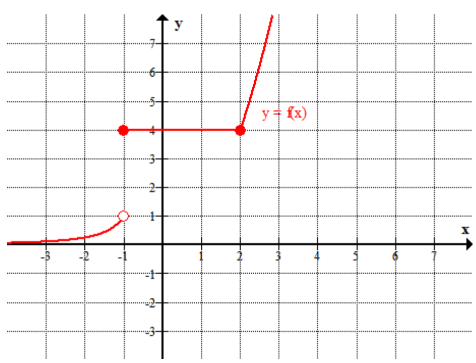
$$f(x)=\begin{cases}\dfrac{1}{x^2} & if\ \ x<-1\\ 4 & if\ \ -1\le x<2\\ x^2 & if\ \ x\ge 2\end{cases}$$

Solution: This function uses three formulas : $1/x^2$ for x < -1; 4 for x in [-1,2); and x^2 for $x \geq 2$

Therefore the graph of f will consist of parts of the graphs of $y = 1/x^2$, $y = 4$, $y = x^2$.



After erasing the dashed lines, we get the graph of f(x)



2.5 Graphing using transformations

Graphs of certain functions can be obtained by graphing one of the basic functions and performing geometric transformations on it.

Shift up and down

Suppose c is a positive real number, c > 0. If the graph of a function g(x) is given, then to obtain the graph of

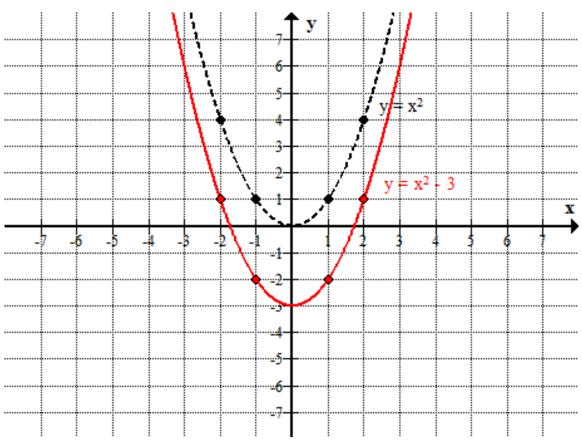
f(x) = g(x) + c

shift the graph of g(x) up by c units.
The graph of f(x) will have the same shape as that of g(x) but it will be in a different place.
Each point on the graph of g(x) will be moved vertically up. If (a,b) is on the graph of g(x) then (a, b + c) will be on the graph of f(x).

f(x) = g(x) – c

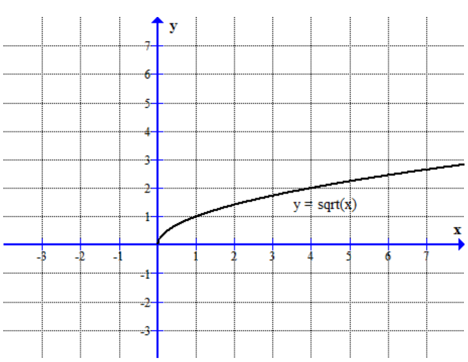
shift the graph of g(x) down by c units.
The graph of f(x) will have the same shape as that of g(x) but it will be in a different place.
Each point on the graph of g(x) will be moved vertically down. If (a,b) is on the graph of g(x) then (a,b-c) will be on the graph of f(x).

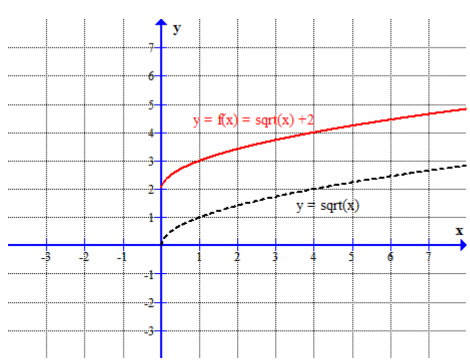
Example: Use transformations to graph $f(x) = x^2 - 3$.
Note that this function is of the form f(x) = g(x) – c, where $g(x) = x^2$ and c = 3. Therefore, the graph of f(x) is obtained by shifting the graph of $y = x^2$ (which we know) by 3 units down.



Example: Use transformations to graph $f(x) = \sqrt{x} + 2$

Solution: note first that this function is of the form f(x) = g(x) + c, where $g(x) = \sqrt{x}$ and c = 2.
Therefore the graph of f is obtained by shifting the graph of $y = \sqrt{x}$ (which we know) by 2 units up





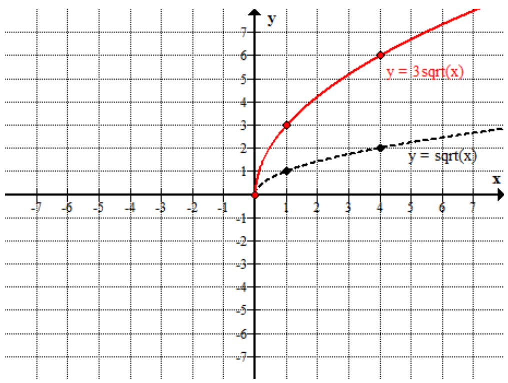
Vertical stretch and compression

If the graph of a function g(x) is given, then to obtain the graph of **f(x) = c g(x)** , c > 0

stretch the graph of g(x) vertically, if c > 1 and compress vertically when 0 < c <1.

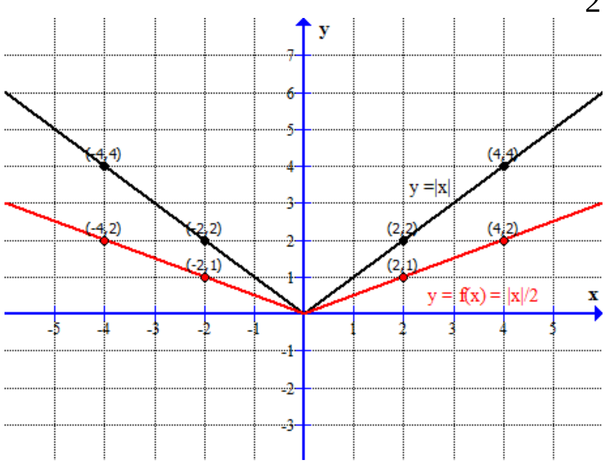
If the point (a,b) is on the graph of g(x) then (a, cb) is on the graph of f(x). The x-intercepts are not affected by this transformation, since y coordinate of an x-intercept is 0.

Example: Use transformations to graph f(x) = $3\sqrt{x}$.
Note first that this function is of the form f(x) = cg(x) , where g(x) = $\sqrt{x}$ and c = 3. Therefore, we obtain the graph of f(x) = $3\sqrt{x}$ by vertically stretching the graph of y = $\sqrt{x}$. If a point (a,b) is on the graph of y = $\sqrt{x}$, then the point (a, 3b) is on the graph of y = $3\sqrt{x}$



Example: Use transformations to graph $f(x) = \frac{1}{2}|x|$

Solution: Note first that the this function is of the form f(x) = c g(x), where c = ½ and g(x) = |x|. Therefore, we obtain the graph of $f(x) = \frac{1}{2}|x|$ by vertically compressing the graph of y = |x|. If a point (a,b) is on the graph of y = |x|, then the point (a, ½ b) is on the graph of $y = \frac{1}{2}|x|$



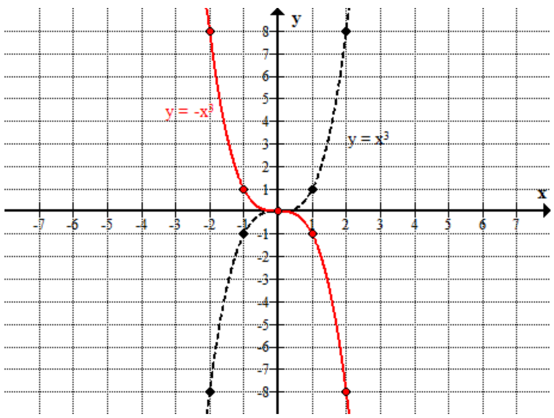
Reflections about the x-axis

If the graph of a function g(x) is given then to obtain the graph of

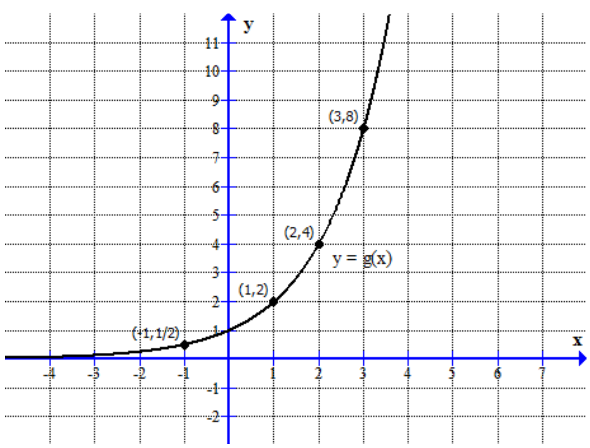
f(x) = - g(x) reflect the graph of g(x) about the x-axis. If a point (a,b) is on the graph of g(x) then (a,-b) is on the graph of f(x) . The graphs of *f* (x)and *g(x)* are symmetric with respect to the x-axis. The x-intercepts will remain intact.

Example: The graph of a function g(x) is given. Sketch the graph of $f(x) = - x^3$.

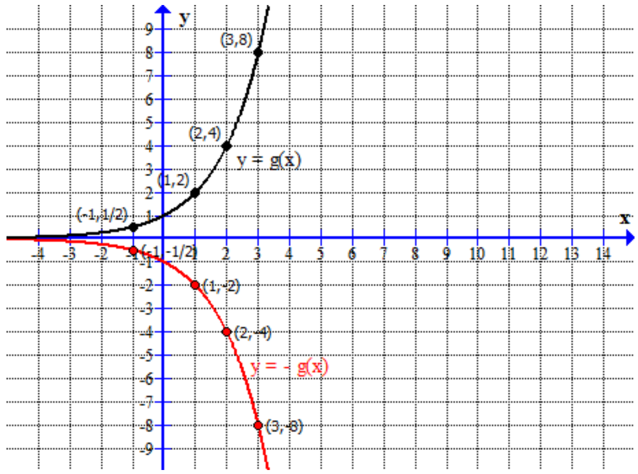
Note first that the this function is of the form f(x) = - g(x), where $g(x) = x^3$. Therefore, we obtain the graph of $f(x) = - x^3$ by reflecting the graph of $y = x^3$ about the x-axis. If a point (a,b) is on the graph of $y = x^3$, then the point (a, - b) is on the graph of $y = - x^3$.



Example: Given the graph of a function g(x). Sketch the graph of f(x) = - g(x).



Solution: To obtain the graph of f(x) we must reflect the graph of g(x) about the x-axis.
If a point (a,b) is on the graph of g(x) then the point (a,-b) will be on the graph of f(x)



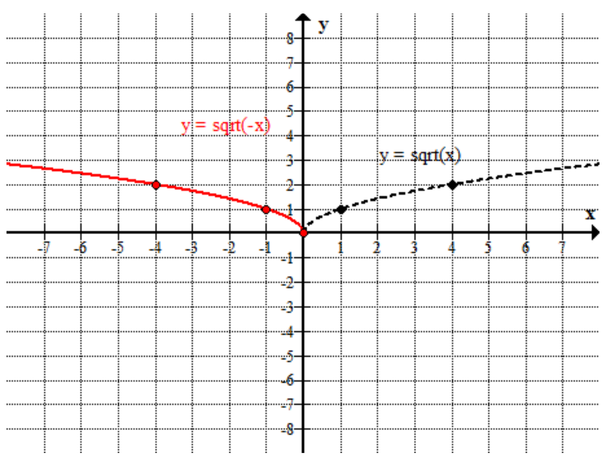
Reflections about the y-axis

If the graph of a function g(x) is given then to obtain the graph of

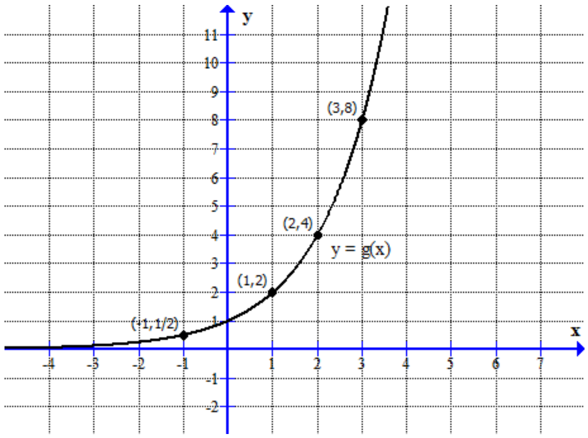
f(x) = g(-x) reflect the graph of g(x) about the y-axis. If a point (x,y) is on the graph of g(x) then (-x, y) is on the graph of f(x). The graphs of f(x) and g(x) are symmetric about the y-axis. The y- intercept remains intact.

Example: Sketch the graph of $f(x) = \sqrt{-x}$.

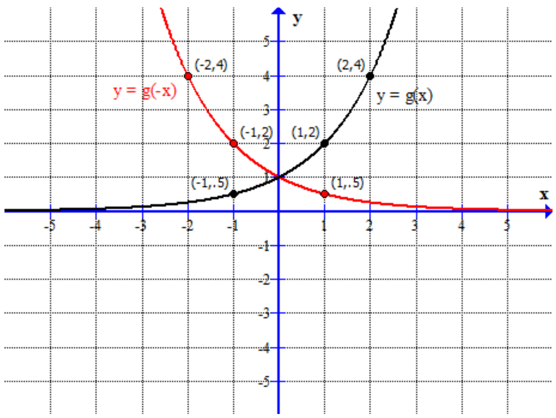
Note first that this function is of the form f(x) = g(-x), where g(x) = $\sqrt{x}$. Therefore, to graph f(x) we must reflect the graph of g(x) about the y-axis. If a point (a,b) is on the graph of g(x) then (-a,b) will be on the graph of g(-x).



Example: The graph of a function g(x) is given below. Sketch the graph of f(x) = g(-x).



Solution: Since f(x) = g(-x), the graph of f(x) is a reflection of the graph of g(x) about the y-axis. If a point (a,b) is on the graph of g(x) , then the point (-a, b) will be on the graph of g(-x) = f(x)



Shift to the left or right

If the graph of g(x) is given then to obtain the graph of

f(x) = g(x + c)

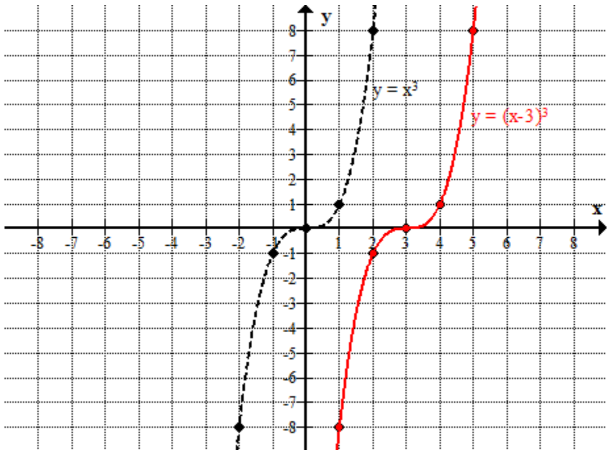
Shift the graph of g(x) horizontally to the left by c units. Each point on the graph of g(x) will be moved horizontally c units to the left If (a,b) is on the graph of f then (a-c, b) will be on the graph of f(x).

f(x) = g(x - c)

Shift the graph of g(x) horizontally to the right by c units. Each point on the graph of g(x) will be moved horizontally c units to the right. If (a,b) is on the graph of f then (a +c, b) will be on the graph of f(x).

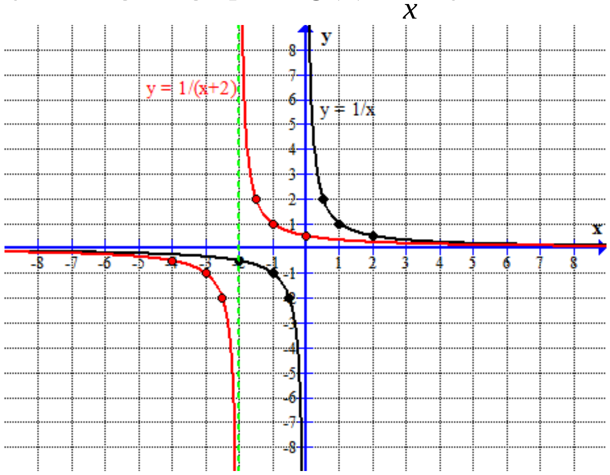
Example: Sketch the graph of f(x) = $(x-3)^3$.

Note first that f(x) is of the form f(x) = g(x-c), where g(x) = x^3 and c = 3. Therefore, the graph of f(x) is obtained by shifting the graph of y = x^3 by 3 units to the right.



Example: Sketch the graph of f(x) = $\frac{1}{x+2}$

Solution: Note that f(x) is of the form y = g(x + c), where $g(x) = \frac{1}{x}$ and c = 2. Therefore, the graph of f(x) is obtained by shifting the graph of $g(x) = \frac{1}{x}$ by 2 units to the left.



Notice that since the graph of $g(x) = \frac{1}{x}$ has a vertical asymptote, it too has to be shifted to the left by 2 units. The same has to be done to a horizontal asymptote, but shifting a horizontal line to the left(or right) does not change the position of that line.

Horizontal stretch or compression

If the graph of a function g(x) is given, them to obtain the graph of

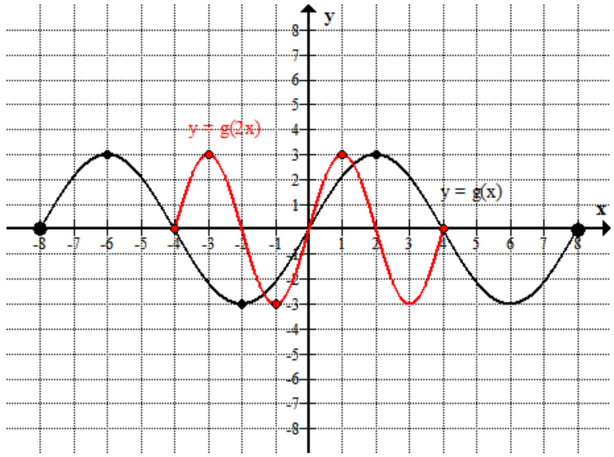
f(x) = g(cx)

compress the graph of g(x) horizontally, if c > 1 and stretch it horizontally if 0 < c < 1. If (a,b) is on the graph of g(x) then $\left(\frac{1}{c}a, b\right)$ will be on the graph of f(x)

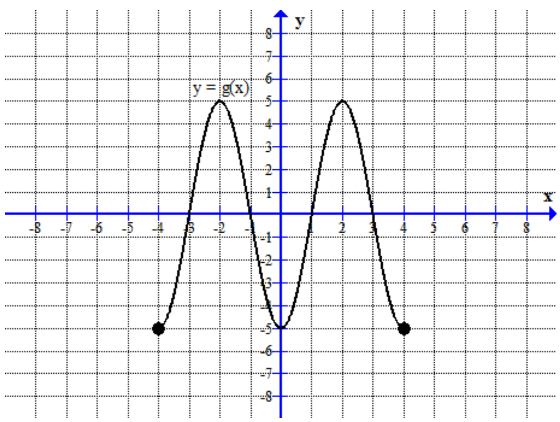
Example: The graph of a function g(x) is given below. Sketch the graph of f(x) = g(2x).

Note first that function f(x) is of the form y = g(cx), where c = 2. Therefore, to graph f(x) we must compress the graph horizontally by a factor of 2. If a point (a,b) is on the graph of g(x) then the point

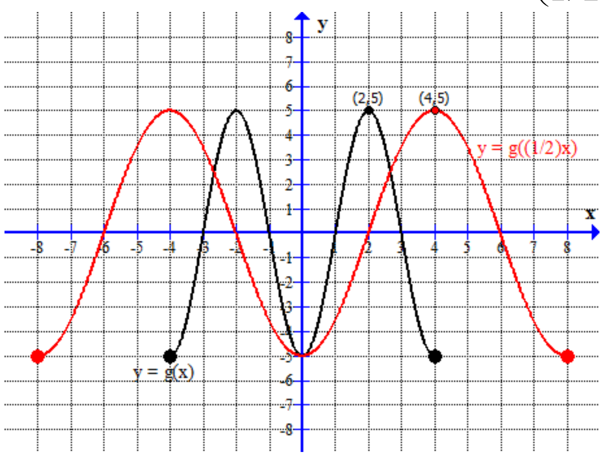
$\left(\frac{1}{2}a, b\right)$ is on the graph of f(x).



Example: The graph of a function g(x) is given below. Sketch the graph of f(x) = $g\left(\frac{1}{2}x\right)$.



Solution: The graph of f(x) is obtained by stretching the graph of g(x) horizontally by a factor of 2, that is, if a point (a,b) is on the graph of g(x) , then the point $\left(\frac{1}{1/2}a, b\right) = (2a, b)$ is on the graph of f(x)



Sequence of Transformations

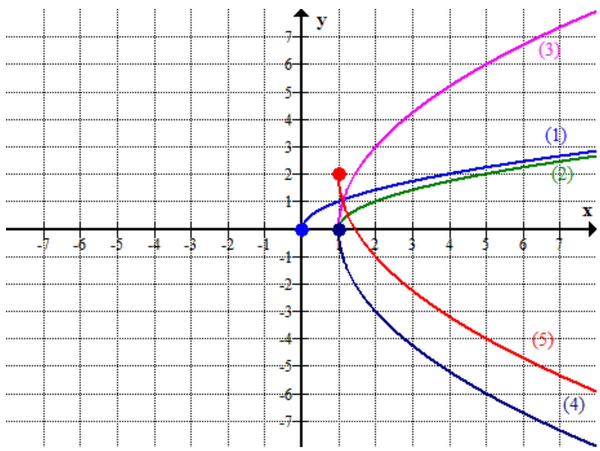
Often to graph a function we must perform multiple transformations. Though order of some transformations can be switched, you can use the following guidelines to graph f(x) = -Ag(-B x+ C) + D:

1) Graph y = g(x)
2) Graph y = g(x+C) (shift left if C >0 and right when C < 0)
3) Graph y = g(Bx + C) (horizontal stretch when 0<B<1; horizontal compression when B >1)
4) Graph y = g(-Bx +C) (reflection about the y-axis)
5) Graph y = Ag(-Bx+C) (vertical stretch when A >1; vertical compression when 0 <A< 1)
6) Graph y = -Ag(-Bx+C) (reflection about the x-axis)
7) Graph y = -Ag(-Bx+C)+D (shift up when D >0; shift down when C < 0)

Example: Use transformations to graph f(x) = $-3\sqrt{x-1}+2$

This function is the transformation of g (x) = $\sqrt{x}$. The order of transformations is as follows

1) Graph y = $\sqrt{x}$ (basic function) (blue)
2) Graph y = $\sqrt{x-1}$ (shift right by 1) (green)
3) Graph y = $3\sqrt{x-1}$ (vertical stretch by a factor of 3) (pink)
4) Graph y = $-3\sqrt{x-1}$ (reflection about the x- axis) (dark blue)
5) Graph y = $-3\sqrt{x-1}+2$ (shift up by 2) (red)

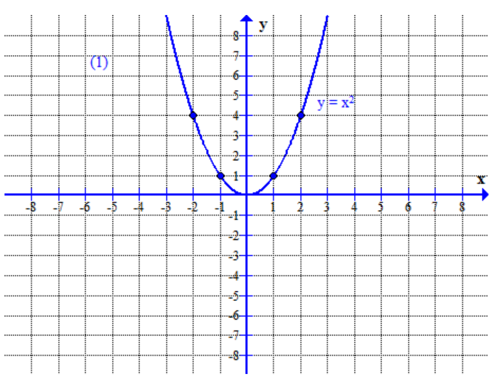


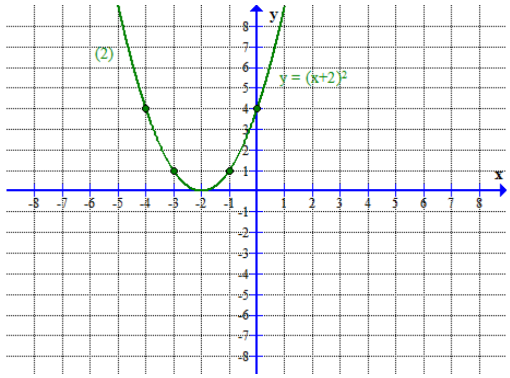
Example: Use transformations to sketch the graph of f(x) = $3(-x+2)^2$ -1. Plot at least three points on the graph of the basic function and use them to perform the transformations. Draw the transformations one at a time.

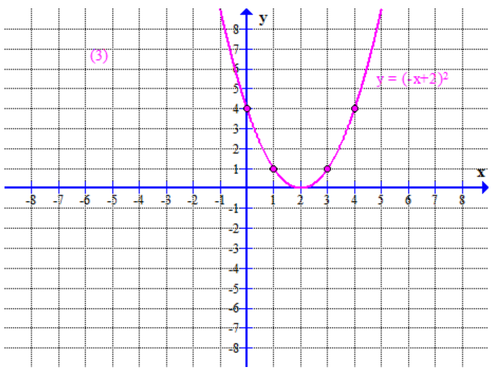
Solution: The order of transformations is as follows;

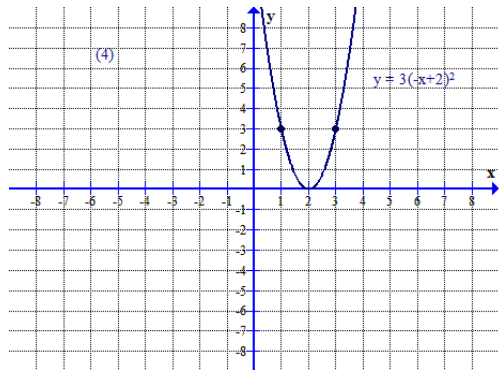
1) $y = x^2$ (basic function)
2) $y = (x + 2)^2$ (shift left by 2)
3) $y = (-x + 2)^2$ (reflect about the y-axis)
4) $y = 3(-x + 2)^2$ (stretch vertically by a factor of 3)
5) $y = 3(-x +2)^2$ -1 (Shift down by 1)

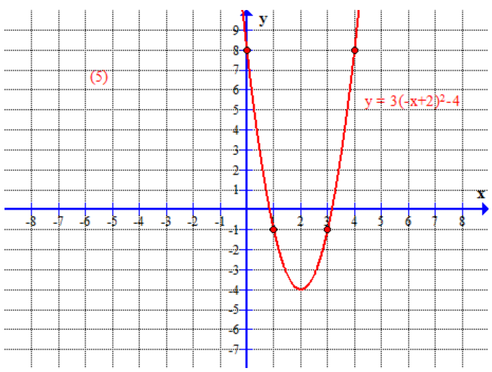
Below are the transformations performed one at a time.











2.6 Operations on functions

Suppose two functions f and g are given and their domains are Df and Dg respectively.
We can form new functions using f and g:
The **sum**, f + g, is the function defined as (f+g)(x) = f(x) + g(x)
The **difference**, f – g, is the function defined as (f- g)(x) = f(x) – g(x)
The **product**, f·g, is a function defined as (f·g)(x) = f(x)·g(x)

The **domain** of these functions is the common part (intersection) of the domains Df and Dg (denoted Df $\cap$ Dg), since we can only add/subtract/multiply f(x) and g(x) only when they exist.

We can also form the **quotient**, f/g, which is defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The domain of f/g is the intersection of Df $\cap$ Dg except for those values x for which g(x) = 0. We will write this as

$$D(f/g)= \{x|x \in Df \cap Dg \text{ and } g(x) \neq 0\}$$

Example: Suppose that $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{x}$. Find a) (f-g) (4), b) (f/g)(x) and its domain.

a) (f-g)(4) = f(4) – g(4) = $\sqrt{4}-\frac{1}{4}=2-\frac{1}{4}=\frac{7}{4}$

b) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt{x}}{\frac{1}{x}}=x\sqrt{x}$

Since Df = [0, +∞), Dg = {x|x ≠ 0}= (-∞,0) ∪ (0, +∞), then Df ∩ Dg = (0, +∞), since g(x) = 0 has no solutions, there is nothing to exclude form Df∩Dg and D(f/g) = Df∩Dg = (0, +∞)

Example: Let $f(x)=\frac{2x+1}{x-3}$ and $g(x)=\frac{\sqrt{x}}{x+1}$. Find a) (f + g) (1), b) (f·g)(x) and its domain

Solution :

a) (f+ g)(1) = f(1) + g(1) = $\frac{2\cdot1+1}{1-3}-\frac{\sqrt{1}}{1+1}=\frac{3}{-2}-\frac{1}{2}=-2$

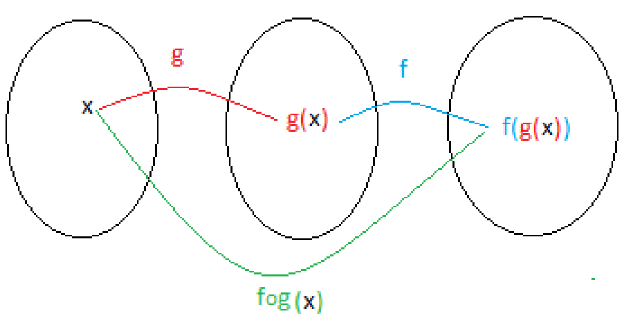
b) $(f\cdot g)(x)=f(x)\cdot g(x)=\frac{2x+1}{x-3}\cdot\frac{\sqrt{x}}{x+1}=\frac{(2x+1)\sqrt{x}}{(x-3)(x+1)}$

Here, Df = {x|x ≠ 3}= (-∞,3) ∪ (3, +∞) and Dg = {x| x ≥ 0 and x ≠ -1}= [0,+ ∞). Therefore, Df ∩ Dg = [0,3) ∪ (3, +∞) and the domain D (f·g) = [0,3) ∪ (3, +∞)

The **composition of f and g**, denoted $f\circ g$, and read g composed with f or " f circle g", is a function defined as

$$(f\circ g)(x)=f(g(x))$$

Composition is an action of two functions in a succession. If x is an input, then g acts on x first, producing the output g(x), which becomes an input for f and produces the output f(g(x)).



The domain of $f\circ g$ is the set of all values x that are in the domain of g for which g(x) is in the domain of f.
We will write D($f\circ g$)={x | $x\in Dg$ *and* $g(x)\in Df$ } (the symbol ∈ means "belongs to")

Example: Find the composition $f\circ g$ and its domain, if $f(x)=\frac{2x+3}{x-4}$ and $g(x)=\frac{2}{x}$

Since ($f\circ g$)(x) = f(g(x)), to find ($f\circ g$)(x), just replace every x in the formula for f(x) by the formula for g(x)

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x}\right) = \frac{2\left(\frac{2}{x}\right)+3}{\left(\frac{2}{x}\right)-4} = \frac{\frac{4}{x}+3}{\frac{2}{x}-4} = \frac{\left(\frac{4}{x}+3\right)\cdot x}{\left(\frac{2}{x}-4\right)\cdot x} = \frac{4+3x}{2-4x}$$

To find the domain:

1) Find the domain of g
   Dg = {x|x≠0}
2) Find the domain of f
   Df = {x| x ≠ 4}
3) Find x's for which g(x) is in the domain of f
   g(x) =2/x
   we want g(x) to be in the domain of f, that is we want g(x) ≠ 4.
   2/x ≠ 4 means 2 ≠ 4x or x ≠1/2
4) Find the intersection (common part) of sets found in 1) and 3)
   D($f \circ g$)={x|x≠0, ½ }

Another way to find the domain is to find the domain of the expression obtained immediately after substituting *g* into *f*. In the example, that expression is

$$\frac{\frac{4}{x}+3}{\frac{2}{x}-4}$$

Determine the domain of this expression .

Here x cannot be zero (x≠ 0), because it would make the "small" fractions $\left(\frac{4}{x} \; and \; \frac{2}{x}\right)$ undefined. But also, the main denominator $\frac{2}{x}-4$ cannot be zero. When is it zero?

$$\frac{2}{x}-4=0$$

$$\frac{2}{x}=4$$

$$4x=2$$

$$x=\frac{1}{2}$$

Therefore x must be different than ½ (x ≠ ½) .
Altogether D (fog) = {x|x ≠0, ½ }

Example: Find the composition $f \circ g$ and its domain, if $f(x)=x^2-3x+2$ and $g(x)=\sqrt{x+2}$

Solution: First we find the composition $f \circ g$

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x+2}\right) = (\sqrt{x+2})^2 - 3(\sqrt{x+2}) + 2 = x+2-3\sqrt{x+2}+2 = x+4-3\sqrt{x+2}$$

For the domain, consider the expression obtained immediately after substituting g(x) into f, which is

$$(\sqrt{x+2})^2 - 3(\sqrt{x+2}) + 2$$

This expression is defined when $x+2 \geq 0$ or $x \geq -2$. Therefore, $D(f \circ g) = \{x \mid x \geq -2\} = [-2, +\infty)$

De-composing a function

Often, it is useful to find two functions whose composition yields a given function.

Example: Find two functions f and g so that $(f \circ g)(x) = \sqrt{x+4}$.

We are looking for two functions f and g such that $f(g(x)) = \sqrt{x+4}$

Think in the following way: starting with any value of x what **two** actions have to be applied to x in order to obtain the given value $\sqrt{x+4}$.

In this example, starting with x we must compute x + 4 first (1st action/function) and then take the square root (2nd action/function).

Therefore g(x) = x + 4 (first action) and f(x) = $\sqrt{x}$ (second action)

It is often helpful to read the formula out loud and translate the phrase into formulas for f(x) and g(x)

(i) Read: *the square root of x plus 4* .

(ii) Divide the phrase into two groups, one before the word *of* (the square root) and one after (x plus 4). The two groups correspond to two functions, f and g, respectively.
Translate the phrases into formulas
f: The square root--- f(x) = $\sqrt{x}$
g: x plus 4 ----g(x) = x+4

This method works perfectly for all functions except the powers, like $(x+4)^3$. But, if you learn to read this expression as "the third power of x plus 4" rather than " x+4 to the third power", you will have no problem with this function either.

Example: Find two functions f and g so that $(f \circ g)(x) = h(x)$, where $h(x) = |x^2 + 4|$. None of the functions can be the identity.

(i) Read: absolute value of x squared plus 4

(ii) Divide absolute value of x squared plus 4

(iii) Define f(x) = |x|, g(x) = $x^2 + 4$

Example: Find two functions f and g so that $(f \circ g)(x) = h(x)$, where h(x) = $\sqrt{3x-2}$. None of the functions can be the identity.

Solution: Read the formula: the square root of 3x -2. Therefore, f(x) = $\sqrt{x}$ (square root function) and g(x) = 3x-2

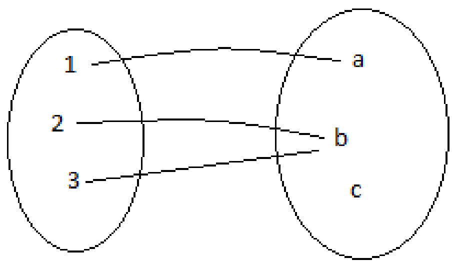
2.7 Inverse functions

One to one functions

A function f is **one to one** if any two different inputs (x values) correspond to two different outputs (y values). That is, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. We can rephrase this statement and say that f is one to one if whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$, which means each value of y is attained only once.

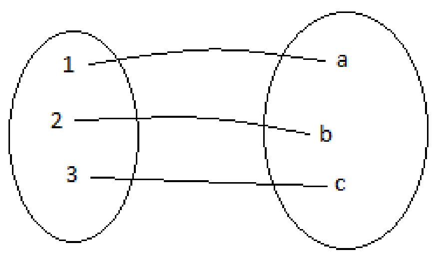
Example: Consider two functions defined by a diagram:

a)



This is a function, but **not one to one** since different inputs (2 and 3) correspond to the same output (b)

b)

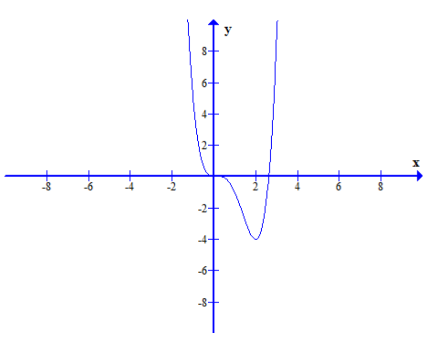


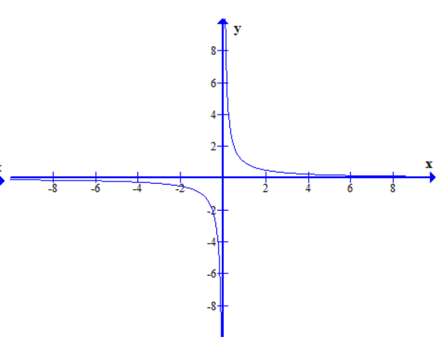
This is a **one to one** function since no two inputs correspond to the same output.

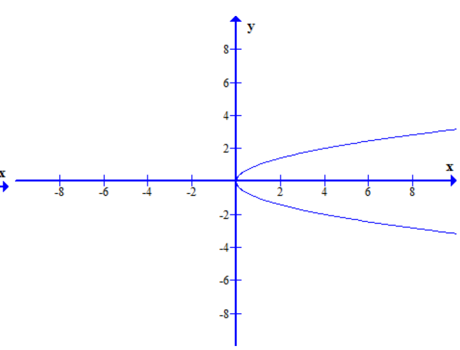
It is easy to recognize a one to one function, if its graph is given. A one to one function will pass the Horizontal Line Test

Horizontal Line Test:
If every horizontal line intersects the **graph of a function** f at most at one point, then f is one to one.

Example : Which of the following graphs represent a one to one function?







Not a one to one

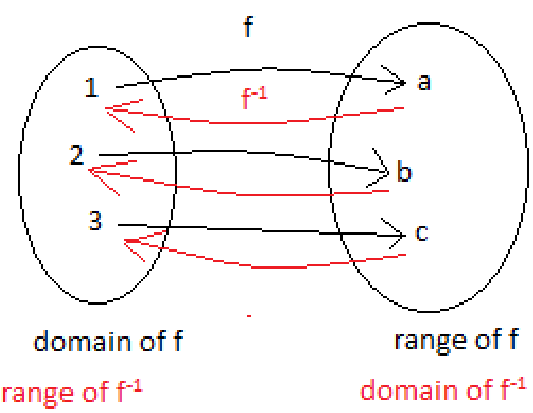
one to one

not a function

Inverse functions - definition

Suppose f(x) = x^2. For any given value of x there is one, and only one, value of y. For example, if x= 2 then y = 2^2 = 4. We can ask then whether we can determine value of x, if value of y is given. For example, if y = 4, do we know what value of x produced this particular y? The answer here is yes, there are two such values: -2 and 2. So even though y = x^2 was a function, the inverse relation (assigning to a given y value of x) is not a function (if y = 4 is an input, there would be two outputs: -2 and 2). So we can ask, what should we know about a function f to guarantee that such an inverse relation remains a function? We would have to guarantee that there is only one input x that gives y as the output. But this means that function f has to be one to one!

If a function *f* is one to one, it has the inverse, denoted by f^{-1}. If a function f assigns to an x a y (that is f(x) = y), then the inverse f^{-1} will assign to that y, the value x (that is f^{-1} (y) = x). This is illustrated on the diagram below.



Function f = {(1,a), (2,b), (3,c)}
Function f^{-1} = {(a,1), (b,2), (c, 3)}
If (x,y) belongs to f, then (y,x) belongs to f^{-1}

We say that f and g are **inverse functions** if $(f \circ g)(x) = x$ for every x in the domain of g (x) and $(g \circ f)(x) = x$ for every x in the domain of f.

Example: Check whether $f(x) = 3x^2 - 2,\ x \geq 0$ and $g(x) = \sqrt{\frac{x+2}{3}},\ x \geq -2$ are inverse functions.

To check that functions f and g are inverses of each other, we must check that $(f \circ g)(x) = x$ for every x in the domain of g (x) and $(g \circ f)(x) = x$ for every x in the domain of f.

Let x $\geq$ -2. Then $(f \circ g)(x) = f(g(x)) = 3\left(\sqrt{\frac{x+2}{3}}\right)^2 - 2 = 3\left(\frac{x+2}{3}\right) - 2 = x + 2 - 2 = x$.

Let x $\geq$ 0.Then $(g \circ f)(x) = g(f(x)) = \sqrt{\frac{(3x^2-2)+2}{3}} = \sqrt{\frac{3x^2}{3}} = \sqrt{x^2} = |x| = x$.

The two conditions are satisfied, therefore, f and g are inverse functions. We can write $f^{-1}(x) = \sqrt{\frac{x+2}{3}},\ x \geq -2$ or $g^{-1}(x) = 3x^2 - 2,\ x \geq 0$.

The diagram above allows us to state the following **properties of the inverse functions**

1. If y = f(x) then $x = f^{-1}(y)$ or equivalently **if x = f(y), then $y = f^{-1}(x)$**
2. Domain of f^{-1}= range of *f*
   Range of f^{-1} = domain of *f*
3. $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}
   $(f^{-1} \circ f)(x) = x$ for all x in the domain of *f*
   (f^{-1} undoes what *f* did)
4. The graphs of *f* and f^{-1} are symmetric with respect to the line y = x.
   If (x, y) is a point on the graph of *f* , then (y,x) is on the graph of f^{-1}. Points (x,y) and (y,x) are symmetric with respect to the line y = x

How to find the inverse of a one to one function:

(i) Write the equation y = *f*(x)
(ii) Switch x with y to write x = *f*(y)
(iii) Solve the equation in (ii) for y (if possible). The solution is $y = f^{-1}(x)$

Example : Find the inverse of $f(x) = \frac{2x+3}{x-4}$ and its domain and range

(i) $y = \frac{2x+3}{x-4}$

(ii) $x = \frac{2y+3}{y-4}$

(iii) $x(y-4) = 2y + 3$

$xy - 4x = 2y + 3$

$xy - 2y = 3 + 4x$

$y(x-2) = 3 + 4x$

$$y = \frac{4x+3}{x-2}$$

$$f^{-1}(x) = \frac{4x+3}{x-2}$$

To find the domain and the range we use property 2 of inverse functions.

Domain of f^{-1} = $\{x | x \neq 2\} = (-\infty, 2) \cup (2, +\infty)$ = Range of *f*

Domain of *f* = $\{x | x \neq 4\} = (-\infty, 4) \cup (4, +\infty)$ = **Range of f^{-1}**

Example: Find the inverse of $f(x) = \sqrt{2x+1}$ and its domain and range

Solution: Function f(x) is a transformation of $y = \sqrt{x}$, therefore it passes the Horizontal Line test and hence is one to one. Function f(x) has the inverse. To find the inverse:

(i) Write the formula y = f(x) : $y = \sqrt{2x+1}$
(ii) Switch x and y: $x = \sqrt{2y+1}$

(iii) Solve for y: $x^2 = 2y + 1$

$$x^2 - 1 = 2y$$

$$\frac{x^2 - 1}{2} = y$$

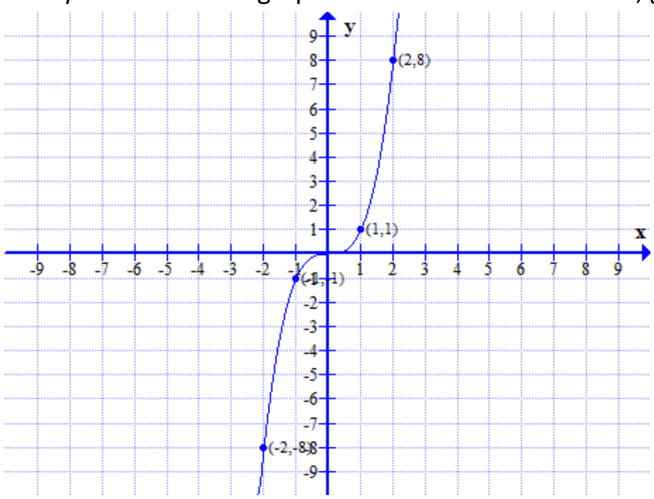
Domain of *f* = {x|x $\geq$ -1/2 } = [-1/2, + ∞) = **Range of *f* $^{-1}$**

Domain of *f* $^{-1}$ = Range of *f* = {y| y $\geq$ 0} = [0, +∞)

Therefore, $f^{-1}(x) = \frac{x^2 - 1}{2}, \quad x \geq 0$.

If a function f(x) has the inverse, then we can use property 4 to obtain the graph of f^{-1} (x), if the graph of f(x) is known. Recall, that the graphs of f(x) and f^{-1}(x) are symmetric with respect to the line y = x. We know that if a point (a,b) is on the graph of f(x), then the point (b,a) is on the graph of f^{-1}(x)

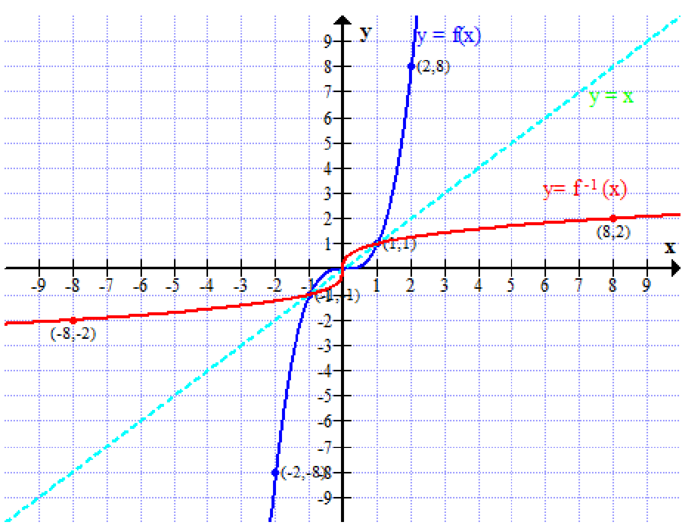
Example: Given the graph of a one to one function *f*, graph its inverse f^{-1}.



Use the property that if (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1} .

Points on the graph of f: (-2, -8), (-1, -1), (0, 0), (1,1), (2,8)

Points on the graph of f^{-1} : (-8,-2), (-1,-1), (0,0), (1,1), (8, 2). Plot the points and sketch the curve. Remember the symmetry.



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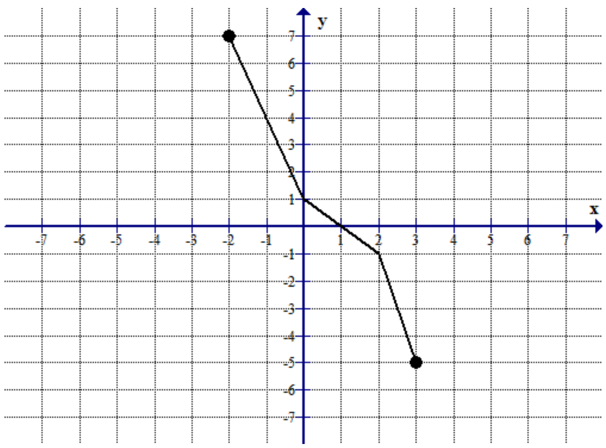

Solution: Points on the graph of a given function f are: (-2,7), (0,1), (1,0), (2,-1), (3,-5).
Points on the graph of f^{-1} are therefore : (7, -2), (1,0), (0,1), (-1,2), (-5,3)