11.1 Sequences

Informally speaking a **sequence** is a succession of numbers, called **terms,** in a definite order

Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

1 is the first term of the sequence; $\frac{1}{2}$ is the second term; $\frac{1}{3}$ is the third terms and so on

Here is the formal definition of a sequence

Definition

A **sequence** is a function whose domain is the set of natural numbers {1,2,3,…}

A sequence of real number has as the range the subset of real numbers. A sequence function assigns to each positive integer n a real number, that we denote a_n (instead of f(n)). Sequences are traditionally denoted by $\{a_n\}$.

A sequence can be defined by

- **Listing** first few **terms**. Enough terms must be listed so it is clear how the terms of the sequence are formed

  Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

  It should be clear here that the next terms are $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \cdots$

- Giving a **formula** for the n-th term

  Example: $a_n = \frac{1}{2^n}, \quad n \geq 1$, or $\left\{\frac{1}{2^n}\right\}_{n\geq 1}$

  Formula allows us to compute any term of the sequence. If we want 10-th term, or a_{10}, we replace n by 10 in the formula $a_n = \frac{1}{2^n}$: $a_{10} = \frac{1}{2^{10}} = \frac{1}{1024}$

- **A recursive formula** where we define explicitly a first (or the first few) term(s) of a sequence and define the n-th term by an equation that involves the preceding terms

  Example: $a_1 = 3$, $a_n = 2a_{n-1} + 5$ for n >1.

  We can compute successive terms by using the formula. So $a_2 = 2a_{2-1}+5 = 2a_1+5 = 2\cdot3+5 = 11$; $a_3 = 2a_{3-1}+5 = 2a_2+5 = 2\cdot11+5 = 27$, and so on.

Example:

Write down the first five terms of the sequence $\{b_n\} = \{n^2-1\}$

$b_1=1^2-1=0$; $b_2 = 2^2 -1=3$; $b_3=3^2-1= 8$; $b_4= 4^2 -1=15$; $b_5 = 5^2-1 = 24$

Therefore the sequence is : 0, 3, 8, 15, 24, ….

Example:

Write the first five terms of the sequence defined recursively as $a_1 = 1$, $a_2 = 2$, $a_n = 2a_{n-1} + a_{n-2}$

We already have a_1 and a_2. We'll use formula $a_n = 2a_{n-1} + a_{n-2}$ for n = 3, 4, 5

$a_3 = 2a_{3-1} + a_{3-2}= 2a_2+a_1=2\cdot2+1=5$

$a_4 = 2a_{4-1} + a_{4-2}=2a_3+a_2= 2\cdot5+2= 12$

$a_5 = 2a_{5-1} + a_{5-2}= 2a_4+a_3 = 2\cdot12+5=29$

Therefore the sequence is: 1, 2, 5, 12, 29, ….

Having a formula for the n-th term of a sequence is the most efficient way to work with a sequence, since we can quickly compute any term of that sequence. For example, if we need 100-th term of a sequence defined recursively we would most likely have to compute all 99 preceding terms. However, not always such a formula is easy to establish.

Example:

Find the formula for the n-th term of a sequence a_n for which the first five terms are

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots$

Note that each term is a fraction with the numerator 1 and the denominators are consecutive powers of 2:

$1 = \frac{1}{2^0}, \quad \frac{1}{2} = \frac{1}{2^1}, \quad \frac{1}{4} = \frac{1}{2^2}, \quad \frac{1}{8} = \frac{1}{2^3}, \quad \cdots$

Therefore $a_1 = 1 = \frac{1}{2^0}, \quad a_2 = \frac{1}{2} = \frac{1}{2^1}, \quad a_3 = \frac{1}{4} = \frac{1}{2^2}, \quad a_4 = \frac{1}{8} = \frac{1}{2^3}, \quad \cdots$. Notice that in the n-th term (first, second, third…) the powers of 2 are n-1. So, we conclude that $a_n = \frac{1}{2^{n-1}}, n = 1,2,3\ldots$. We could simplify the formula, if we started counting from zero: $a_n = \frac{1}{2^n}, n = 0,1,2,3,\ldots$

Example:

Find the formula for the n-th term of a sequence a_n for which the first five terms are: 1, -1, 1, -1, 1 …

In this sequence, the terms alternate the sign. Note that $(-1)^{\text{even power}} = 1$ and $(-1)^{\text{odd power}} = -1$. Since for positive integers, the numbers alternate between odd and even, we see that $\{(-1)^{n-1}\}_{n\geq 1} = \{1, -1, 1, -1, \ldots\}$ and $\{(-1)^n\}_{n\geq 1} = \{-1, 1, -1, 1, \ldots\}$. Therefore, for the given sequence $a_n = (-1)^{n-1}$, n = 1,2,3,…

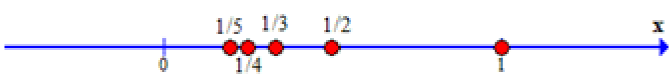
Geometrically a sequence can be illustrated in two ways

- The terms of a sequence can be plotted on the number line
- Points (n, a_n) can be plotted in the coordinate plane and represent the graph of the function (a(n))

Example:

Illustrate the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ on the number line.

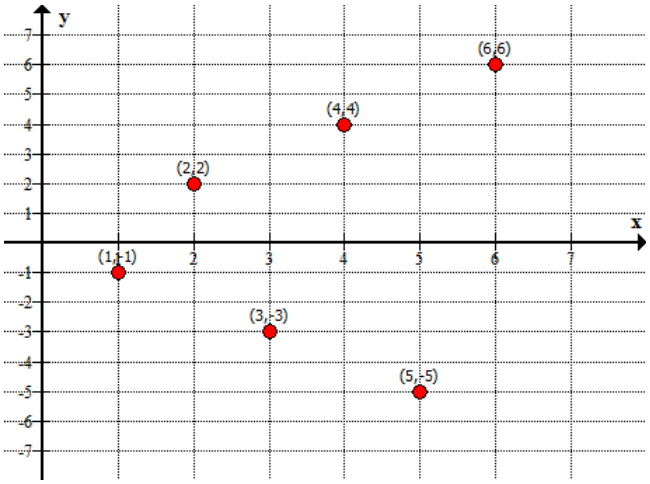
The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ can be represented by a formula $a_n = \frac{1}{n}$, $n \geq 1$. Numbers a_n are plotted on the number line:



Example:

Illustrate the sequence $\{(-1)^n n\}$ in the coordinate plane.

Points $(n, (-1)^n n)$ are plotted in the coordinate plane. The first few points are (1,-1), (2,2), (3,-3), (4,4)…



Sigma notation

If a sequence $\{a_n\}$ is given then the sum of the first n **consecutive** terms of such sequence is $a_1 + a_2 + a_3 + \ldots a_n$ are shortly written using sigma notation. Sigma is a Greek letter which corresponds to S in English alphabet. It is written as $\sum$. So the sum above is written as

$$a_1 + a_2 + a_3 + \ldots a_n = \sum_{k=1}^{n} a_k$$

In the notation $\sum_{k=1}^{n} a_k$, $\sum a_k$ indicates that we must add the consecutive terms of a sequence $\{a_k\}$; the "k=1"on the bottom of $\sum$ tells us to start with the term a_k, where k = 1 (that is with the term a_1); the "n" on the top of $\sum$, tells us to stop adding when we reach term a_n. For example, $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$. We can use sigma notation to indicate any sum of consecutive terms of a sequence . For example,

$$\sum_{k=5}^{10} a_k = a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$$

Example:

Write out (expand) the sum $\sum_{k=1}^{5}(2k+3)$ and find its value.

We must add consecutive terms of the sequence a_k = 2k+3, starting with k=1 and ending with k = 5

$$\sum_{k=1}^{5}(2k+3) = (2\cdot 1+3)+(2\cdot 2+3)+(2\cdot 3+3)+(2\cdot 4+3)+(2\cdot 5+3) = 5+7+9+11+13 = 45$$

Example:

Express the sum using sigma notation $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \frac{32}{243} - \frac{64}{729}$

First note that $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \frac{32}{243} - \frac{64}{729} = \frac{2}{3} + \left(-\frac{4}{9}\right) + \frac{8}{27} + \left(-\frac{16}{81}\right) + \frac{32}{243} + \left(-\frac{64}{729}\right)$

Therefore we must add 6 terms of the sequence:

$$\frac{2}{3}, -\frac{4}{9}, \frac{8}{27}, -\frac{16}{81}, \frac{32}{243}, -\frac{64}{729}, \ldots$$

First we find the formula for n-th term of this sequence. Note that the numerators are powers of 2 and the denominators are powers of 3. The sign alternates, so there must be $(-1)^n$ or $(-1)^{n+1}$ present. Therefore,

$a_n = (-1)^{n+1}\frac{2^n}{3^n} = (-1)^{n+1}\left(\frac{2}{3}\right)^n, n \geq 1$. We use this term within sigma:

$$\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\frac{16}{81}+\frac{32}{243}-\frac{64}{729} = \frac{2}{3}+\left(-\frac{4}{9}\right)+\frac{8}{27}+\left(-\frac{16}{81}\right)+\frac{32}{243}+\left(-\frac{64}{729}\right) = \sum_{k=1}^{6}(-1)^{k+1}\left(\frac{2}{3}\right)^k$$

Properties of sigma notation

If $\{a_k\}$ and $\{b_k\}$ are two sequences of real numbers then

(I) $\sum_{k=n}^{m} ca_k = c\sum_{k=n}^{m} a_k$

(II) $\sum_{k=n}^{m}(a_k \pm b_k) = \sum_{k=n}^{m} a_k \pm \sum_{k=n}^{m} b_k$

Property (I) is simply generalized distribution property, and property (II) follows from the fact that when adding numbers we can group them in any way we like (commutative and associative properties of addition)

Example:

Suppose that $\{a_k\}$ and $\{b_k\}$ are two sequences such that $\sum_{k=1}^{15} a_k = 9, \sum_{k=1}^{15} b_k = -6$

Then $\sum_{k=1}^{15}(3a_k - 2b_k) = \sum_{k=1}^{15} 3a_k - \sum_{k=1}^{15} 2b_k = 3\sum_{k=1}^{15} a_k - 2\sum_{k=1}^{15} b_k = 3\cdot 9 - 2(-6) = 39$

Special Sums

(A) $\sum_{k=1}^{n} c = \underbrace{c+c+c+...+c}_{n\ terms} = n\cdot c$, where c is any real number

(B) $\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{n(n+1)}{2}$

(C) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$

These formulas can be proven in an elementary way. We omit the proof here.

Properties of sigma and special sum allow us to compute large sums with a relative ease.

Example:

Find the value of the sum $\sum_{k=1}^{184}(3k+5)$

$$\sum_{k=1}^{184}(3k+5) = \sum_{k=1}^{184} 3k + \sum_{k=1}^{184} 5 = 3\sum_{k=1}^{184} k + 5\cdot 184 = 3\cdot\frac{184(184+1)}{2} + 920 = 3\cdot 92\cdot 185 + 920 = 51{,}980$$

11.2 Arithmetic Sequences

A sequence $\{a_n\}$ is **arithmetic**, if the difference of two consecutive terms is always the same, that is for any $n > 1$, $a_n - a_{n-1} = d$ =constant (independent of n). The difference, d, is called the common difference.

Note that the terms of an arithmetic sequence are formed by adding the common difference, d, to the previous term: $a_n = a_{n-1} + d$. Therefore, if a_1 is the first term of the arithmetic sequence, then,

$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$
$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$
$$\ldots$$

Example:

Determine whether given sequence is arithmetic.

a) $\{a_n\} = \left\{\frac{n+1}{n+2}\right\}$, $n \geq 1$

We must show that $a_n - a_{n-1}$ is constant, for n > 1. Since $a_n = \frac{n+1}{n+2}$, then $a_{n-1} = \frac{(n-1)+1}{(n-1)+2} = \frac{n}{n+1}$

Therefore $a_n - a_{n-1} = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - n(n+2)}{(n+2)(n+1)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)}$.

The difference $a_n - a_{n-1}$ depends on n, so it is not a constant, and therefore the sequence is not arithmetic.

b) $\{b_n\} = \{3n+1\}$, $n \geq 1$

We must show that $b_n - b_{n-1}$ is a constant, for n > 1. Since $b_n = 3n+1$, then $b_{n-1} = 3(n-1) + 1 = 3n - 2$

Therefore $b_n - b_{n-1} = (3n+1) - (3n-2) = 3n + 1 - 3n + 2 = 3$. Since $b_n - b_{n-1}$ is a constant, sequence is arithmetic.

Theorem

If $\{a_n\}$ is an arithmetic sequence with the first term a_1 and common difference d, then

$a_n = a_1 + (n-1)d$, $n \geq 1$

and

$$a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k = \frac{a_1 + a_n}{2} \cdot n$$

Example:

Find the 26-th term of an arithmetic sequence with the first term ½ and the common difference -1.

Denote the sequence $\{a_n\}$. Then a_1= ½ and d = -1. Since the sequence is arithmetic,

$a_n = a_1 + (n-1)d = \frac{1}{2} + (n-1)(-1) = -n + \frac{3}{2}$ for $n \geq 1$. Therefore , when n = 26, we get

$a_{26} = -26 + \frac{3}{2} = -\frac{49}{2}$

Example:

Find the 200-th term of the arithmetic sequence $\sqrt{3}, 3\sqrt{3}, 5\sqrt{3}, 7\sqrt{3}, \ldots$

Denote the sequence $\{a_n\}$. Since the sequence is arithmetic, $a_1 = \sqrt{3}, d = a_2 - a_1 = 3\sqrt{3} - \sqrt{3} = 2\sqrt{3}$.

Therefore, the n-th term of the sequence is

$a_n = a_1 + (n-1)d = \sqrt{3} + (n-1)(2\sqrt{3}) = 2n\sqrt{3} - \sqrt{3} = (2n-1)\sqrt{3}$

And consequently, $a_{200} = (2 \cdot 200 - 1)\sqrt{3} = 399\sqrt{3}$

Example:

Find the first term and the common difference, d, for an arithmetic sequence whose 4^{th} term is 3 and 20^{th} term is 35. Give the recursive formula for the sequence and write the formula for the n-th term

If $\{a_n\}$ is the sequence, then $a_4 = 3 \quad and \; a_{20} = 35$. Since the sequence is arithmetic, then $a_n = a_1 + (n-1)d$ for $n \geq 1$.

Therefore,

$$3 = a_4 = a_1 + 3d$$
$$35 = a_{20} = a_1 + 19d$$

Solving the system

$$a_1 + 3d = 3$$
$$a_1 + 19d = 35$$

for a_1 and d, we get d= 2 and a_1 = -3.

Therefore the n-th term of this sequence is $a_n = a_1 + (n-1)d = -3 + (n-1) \cdot 2 = 2n - 5$ and the recursive formula is

$$a_1 = -3$$
$$a_n = a_{n-1} + d = a_{n-1} + 2$$

Example:

Find the following sum of the arithmetic sequence

31+ 34+ 37 + … + 88

Denote the sequence $\{a_n\}$. Since the sequence is arithmetic, $a_1 = 31, d = a_2 - a_1 = 34 - 31 = 3$.

Then $a_n = a_1 + (n-1)d = 31 + (n-1) \cdot 3 = 3n + 28$. *To apply the formula for the sum of the first n terms of an arithmetic sequence (* $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k = \frac{a_1 + a_n}{2} \cdot n$), we must find out how many terms of given

sequence $\{a_n\}$ are added or the value of n. Since the last term in the given sum is 88, then a_n = 88 and therefore, 3n+28 = 88. Solving this equation gives n= 20.
Thus we get

$$31+34+37+\ldots+88=\frac{a_1+a_n}{2}\cdot n=\frac{31+88}{2}\cdot 20=1190$$

Example :

Find the sum $\sum_{n=1}^{80}\left(\frac{1}{2}n+\frac{1}{3}\right)$

We can use properties and formulas discussed in the previous section to evaluate this sum or we can notice that the sequence $a_n=\frac{1}{2}n+\frac{1}{3}$ is arithmetic and use formula for the first n terms of an arithmetic sequence.

Since the first method is easier, we'll use properties of sigma to evaluate this sum

$$\sum_{n=1}^{80}\left(\frac{1}{2}n+\frac{1}{3}\right)=\frac{1}{2}\sum_{n=1}^{80}n+\sum_{n=1}^{80}\frac{1}{3}=\frac{1}{2}\cdot\frac{80(80+1)}{2}+\frac{1}{3}\cdot 80=20\cdot 81+\frac{80}{3}=\frac{4940}{3}$$

11.3 Geometric sequence

A sequence $\{a_n\}$ is **geometric** , if the ratio of two consecutive terms is always the same nonzero number, that is for any $n > 1$, $\frac{a_n}{a_{n-1}}=r$ =constant (independent of n), $r\neq 0$, $a_1\neq 0$. The ratio r , is called the common ratio.

Note that the terms of a geometric sequence are formed by multiplying the previous term by the common ration r, $a_n=ra_{n-1}$. Therefore, if $a_1\neq 0$, is the first term of the arithmetic sequence, then,

$$a_2=ra_1$$
$$a_3=ra_2=r(ra_1)=a_1r^2$$
$$a_4=ra_3=r\left(a_1r^2\right)=a_1r^3$$
$$\ldots$$

Example:

Determine whether given sequence is geometric.

a) $\{a_n\}=\left\{\frac{n+1}{2^n}\right\}$, $n\geq 1$

We must show that $\frac{a_n}{a_{n-1}}$ is a constant, for n > 1. Since $a_n=\frac{n+1}{2^n}$, then $a_{n-1}=\frac{(n-1)+1}{2^{n-1}}=\frac{n}{2^{n-1}}$. Therefore

$$\frac{a_n}{a_{n-1}}=\frac{\frac{n+1}{2^n}}{\frac{n}{2^{n-1}}}=\frac{(n+1)2^{n-1}}{2^n\cdot n}=\frac{n+1}{2n}$$. The quotient $\frac{a_n}{a_{n-1}}$ depends on n, so it is not a constant, and therefore the sequence is not geometric.

b) $\{b_n\} = \left\{-2\left(\frac{3}{5}\right)^{n+1}\right\}$, $n \geq 1$

We must show that $\frac{b_n}{b_{n-1}}$ is a constant, for n > 1. Since $b_n = -2\left(\frac{3}{5}\right)^{n+1}$, then $b_{n-1} = -2\left(\frac{3}{5}\right)^{(n-1)+1} = -2\left(\frac{3}{5}\right)^{2}$

Therefore $\frac{b_n}{b_{n-1}} = \frac{-2\left(\frac{3}{5}\right)^{n+1}}{-2\left(\frac{3}{5}\right)^{n}} = \frac{3}{5}$. Since $\frac{b_n}{b_{n-1}}$ is a constant, sequence is geometric.

Theorem

If $\{a_n\}$ is a geometric sequence with the first term $a_1 \neq 0$ and common ratio $r \neq 0$, then

$a_n = a_1 r^{n-1}$, $n \geq 1$

and

$$a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k = \begin{cases} a_1 \dfrac{1-r^n}{1-r} & \text{if } r \neq 1 \\ na_1 & \text{if } r = 1 \end{cases}$$

Example:

Find the 8-th term of a geometric sequence with the first term 4 and the common ratio -1/2 .

Denote the sequence $\{a_n\}$. Then a_1= 4 and d = -1/2 . Since the sequence is geometric,

$a_n = a_1 r^{n-1} = 4 \cdot \left(\frac{-1}{2}\right)^{n-1}$ for $n \geq 1$. Therefore , when n = 8, we get $a_8 = 4\left(\frac{-1}{2}\right)^{7} = 4 \cdot \frac{(-1)^7}{2^7} = \frac{-1}{32}$

Example:

Find the 50-th term of the geometric sequence $\sqrt{3}, 3, 3\sqrt{3}, 9, 9\sqrt{3}, \ldots$

Denote the sequence $\{a_n\}$. Since the sequence is geometric, $a_1 = \sqrt{3}$, $r = \frac{a_2}{a_1} = \frac{3}{\sqrt{3}} = \sqrt{3}$.

Therefore, the n-th term of the sequence is $a_n = a_1 r^{n-1} = \sqrt{3} \cdot \left(\sqrt{3}\right)^{n-1}$

And consequently, $a_{50} = \sqrt{3}\left(\sqrt{3}\right)^{50-1} = \left(\sqrt{3}\right)^{50} = 3^{50/2} = 3^{25}$

Example:

Find the first term and the common ratio, r, for a geometric sequence whose 4^{th} term is 15 and 20^{th} term is 45. Give the recursive formula for the sequence and write the formula for the n-th term

If $\{a_n\}$ is the sequence, then $a_4 = 3$ *and* $a_{20} = 35$. Since the sequence is geometric , then $a_n = a_1 r^{n-1}$ for $n \geq 1$.

Therefore,

$$15 = a_4 = a_1 r^3$$
$$45 = a_{20} = a_1 r^{19}$$

To find a_1 and r, we must solve the system

$$a_1 r^3 = 15$$
$$a_1 r^{19} = 45$$

Note that $\frac{45}{15} = \frac{a_1 r^{19}}{a_1 r^3} = r^{16}$, hence, $r^{16} = 3$ and $r = \sqrt[16]{3} = 3^{\frac{1}{16}}$. Then , using the first equation, we get

$$a_1 = \frac{15}{r^3} = \frac{15}{\left(3^{\frac{1}{16}}\right)^3} = \frac{15}{3^{\frac{3}{16}}} = \frac{15}{\sqrt[16]{9}}$$

Therefore the n-th term of this sequence is $a_n = a_1 r^{n-1} = \frac{15}{3^{\frac{3}{16}}} \cdot \left(3^{\frac{1}{16}}\right)^{n-1} = \frac{15 \cdot (3)^{\frac{n-1}{16}}}{3^{\frac{3}{16}}} = 15 \cdot (3)^{\frac{n-1}{16} - \frac{3}{16}} = 15 \cdot (3)^{\frac{n-4}{16}}$

and the recursive formula is

$$a_1 = \frac{15}{3^{\frac{3}{16}}}$$
$$a_n = r a_{n-1} = 3^{\frac{1}{16}} a_{n-1}$$

Example:

Find the following sum of the geometric sequence

$$2 + \frac{6}{5} + \frac{18}{25} + \cdots + 2\left(\frac{3}{5}\right)^{21}$$

Denote the sequence $\{a_n\}$. Since the sequence is geometric, $a_1 = 2,\ r = \frac{a_2}{a_1} = \frac{\frac{6}{5}}{2} = \frac{3}{5}$.

Then $a_n = a_1 r^{n-1} = 2 \cdot \left(\frac{3}{5}\right)^{n-1}$. To apply the formula for the sum of the first n terms of an arithmetic sequence *(* $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k = \begin{cases} a_1 \frac{1-r^n}{1-r} & if\ r \neq 1 \\ n a_1 & if\ r = 1 \end{cases}$), we must find out how many terms of given sequence

$\{a_n\}$ are added that is we need the value of n. Since the last term in the given sum is $2\left(\frac{3}{5}\right)^{21}$, then

$a_n = 2\left(\frac{3}{5}\right)^{21}$ and therefore, $a_n = 2 \cdot \left(\frac{3}{5}\right)^{n-1} = 2\left(\frac{3}{5}\right)^{21}$. Clearly, n-1= 21, that is n = 22

Thus we get

$$2+\frac{6}{5}+\frac{18}{25}+\cdots+2\left(\frac{3}{5}\right)^{21}=a_1\frac{1-r^{22}}{1-r}=2\cdot\frac{1-\left(\frac{3}{5}\right)^{22}}{1-\frac{3}{5}}=5\left(1-\left(\frac{3}{5}\right)^{22}\right)$$

Example :

Find the sum $\sum_{k=1}^{15}4\cdot(3)^{k-1}$

Sequence $a_n = 4(3)^{n-1}$ is geometric since $\frac{a_n}{a_{n-1}}=\frac{4\cdot 3^{n-1}}{4\cdot 3^{(n-1)-1}}=3^{n-1-(n-2)}=3$. Therefore, r = 3 and $a_1 = 4(3)^{1-1} = 4$.

Hence

$$\sum_{k=1}^{15}4\cdot(3)^{k-1}=\sum_{k=1}^{15}a_k=a_1\frac{1-r^{15}}{1-r}=4\cdot\frac{1-3^{15}}{1-3}=-2(1-3^{15})$$

11.5 Binomial Theorem

The Binomial Theorem give the general formula for $(x+a)^n$. Before stating it however, we need to introduce notation.

Definition:

If n is a positive integer, we define **n factorial**, denoted as n!, as

$0! = 1$

$1! = 1$

$n! = 1\cdot 2\cdot 3\cdot\cdots\cdot n$

Binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, $0 \le k \le n$. The symbol $\binom{n}{k}$ is read " n choose k".

Example:

Compute 6*!*

$$6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 720$$

Note that $6! = \underbrace{1\cdot 2\cdot 3\cdot 4}_{4!}\cdot 5\cdot 6 = 4!\cdot 5\cdot 6$

In general, for any k < n, we have $n! = k!\cdot(k+1)(k+2)\cdots n$

Example:

Since 5 < 9, we can rewrite 9! as $9! = 5!\cdot 6\cdot 7\cdot 8\cdot 9$.

Similarly 11! Can be written as $11! = 8!\cdot 9\cdot 10\cdot 11$ or $11! = 5!\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11$

Example:

Compute the following binomial coefficients : $\binom{7}{3}$ and $\binom{5}{0}$

a) $\binom{7}{3} = \frac{7!}{3!\cdot(7-3)!} = \frac{7!}{3!\cdot 4!} = \frac{4!\cdot 5\cdot 6\cdot 7}{3!\cdot 4!} = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3} = 35$

b) $\binom{5}{0} = \frac{5!}{0!\cdot(5-0)!} = \frac{5!}{0!\cdot 5!} = \frac{5!}{1\cdot 5!} = 1$

Here are some properties of binomial coefficients

Theorem

If n, k are positive integers such that k $\leq$ n, then

(i) $\binom{n}{0} = 1$; $\binom{n}{n} = 1$

(ii) $\binom{n}{1} = n$; $\binom{n}{n-1} = n$

(iii) $\binom{n}{k} = \binom{n}{n-k}$

(iv) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Properties (i) –(iii) follow directly from the definition of the binomial coefficient. To show that property (iv) holds we use the definition and algebra to transform the left hand side into the right hand side:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n+1-k)!} + \frac{n!}{k!\cdot(n-k)!} =$$

$$\frac{n!}{(k-1)!(n-k)!(n+1-k)!} + \frac{n!}{(k-1)!\cdot k\cdot(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \left[\frac{1}{n+1-k} + \frac{1}{k}\right] =$$

$$\frac{n!}{(k-1)!(n-k)!}\,\frac{k+n+1-k}{(n+1-k)k} = \frac{n!(n+1)}{(k-1)!(n-k)!(n+1-k)k} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

Example

a) $\binom{5}{3} + \binom{5}{4} = \binom{6}{4}$ (here n = 5 and k = 4)

b) $\binom{7}{2} + \binom{7}{3} = \binom{8}{3}$ (here n = 7 and k = 3)

Binomial Theorem

$$(x+a)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}a^k = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}x^1a^{n-1} + a^n$$

Remarks:

Note that $\sum_{k=0}^{n}\binom{n}{k}x^{n-k}a^k = \binom{n}{0}x^na^0 + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}x^1a^{n-1} + \binom{n}{n}x^{n-n}a^n =$

$$x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}x^1a^{n-1} + a^n$$

Note that in the expansion of $(x+a)^n$, the exponents of x decrease from *n* to 0, while the exponents of *a* increase from 0 to n. The sum of the exponents of x and a in each term is always *n*

Example

Use the Binomial Theorem to expand $(x+2)^5$. Here n = 5 and a = 2.

$$(x+2)^5 = \sum_{k=0}^{5}\binom{5}{k}x^{5-k}2^k = \binom{5}{0}x^{5-0}2^0 + \binom{5}{1}x^{5-1}2^1 + \binom{5}{2}x^{5-2}2^2 + \binom{5}{3}x^{5-3}2^3 + \binom{5}{4}x^{5-4}2^4 + \binom{5}{5}x^{5-5}2^5 =$$

$$x^5 + 5x^4\cdot 2 + 10x^3\cdot 4 + 10x^2\cdot 8 + 5x\cdot 16 + 32 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

Example

Use the Binomial Theorem to expand $(x^2 - y^2)^6$.

First we must re-write the binomial so it resembles the binomial in the Theorem: $(x^2 - y^2)^6 = (x^2 + (-y^2))^6$. So, here n = 6, x is replaced by x^2 and a is replaced by $(-y^2)$

$$(x^2 - y^2)^6 = (x^2 + (-y^2))^6 = \sum_{k=0}^{6}\binom{6}{k}(x^2)^{6-k}(-y^2)^k =$$

$$\binom{6}{0}(x^2)^{6-0}(-y^2)^0 + \binom{6}{1}(x^2)^{6-1}(-y^2)^1 + \binom{6}{2}(x^2)^{6-2}(-y^2)^2 + \binom{6}{3}(x^2)^{6-3}(-y^2)^3 + \binom{6}{4}(x^2)^{6-4}(-y^2)^4 +$$

$$\binom{6}{5}(x^2)^{6-5}(-y^2)^5 + \binom{6}{6}(x^2)^{6-6}(-y^2)^6 = x^{12} + 6x^{10}(-y^2) + 15x^8y^4 + 20x^6(-y^6) + 15x^4y^8 + 6x^2(-y^{10}) + y^{12} =$$

$$x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 6x^2y^{10} + y^{12}$$

Example

Find the coefficient of x^6 in the expansion of $(x+3)^{10}$

By the Binomial Theorem

$$(2x+3)^{10} = \sum_{k=0}^{10}\binom{10}{k}(2x)^{10-k}3^k = \sum_{k=0}^{10}\binom{10}{k}2^{10-k}x^{10-k}3^k = \sum_{k=0}^{10}\binom{10}{k}2^{10-k}3^kx^{10-k}$$

The terms in the expansions are of the form $\binom{10}{k} 2^{10-k} 3^k x^{10-k}$. Since we are looking for the term that contains x^6, we want 10-k to be 6: 10-k = 6. Therefore k = 4. Then the coefficient is $\binom{10}{k} 2^{10-k} 3^k$ with k = 4. Therefore, the coefficient of x^6 is $\binom{10}{4} 2^{10-4} 3^4 = 210 \cdot 2^6 \cdot 3^4 = 210 \cdot 64 \cdot 81 = 1{,}088{,}640$

Coefficients of $(x+a)^N$ can be arranged in a triangular table called **Pascal's triangle**

N = 0						1					
N= 1					1		1				
N=2				1		2		1			
N= 3			1		3		3		1		
N= 4		1		4		6		4		1	
N= 5	1		5		10		10		5		1

⋮

To find the coefficients of $(x+a)^5$ start with a 1, the next coefficient will be the sum of the two coefficients in the row above (shown in red: 5 = 1+4) , next coefficient will be the sum of 4 and 6 (thus 10) , the next (shown in blue) is the sum of 6 and 4, and so on. End the row with a 1 to form a triangular array of numbers.

Using Pascal's triangle and remark about the exponents of x and a, we can quickly write the expansion of binomial $(x+a)^n$ for small values of n. For example,

$(x+a)^5 = 1x^5a^0 + 5x^4a^1 + 10x^3a^2 + 10x^2a^3 + 5x^1a^4 + 1x^0a^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$