# Set Theory - an introduction

Set - a collection of well defined objects
Objects in the set are called elements or members
Sets - denoted by uppercase letters
Elements - denoted by lower case letters
$x \in A$, $x \notin A$

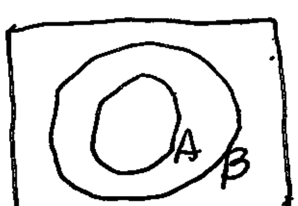
<u>Def.</u> $P = \{1, 2, 3, \ldots\}$ positive integers
$N = \{0, 1, 2, \ldots\}$ natural numbers
$Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ integers
$Q$ = rational numbers = $\{\frac{m}{n}, m, n \in Z, n \neq 0\}$
$\mathbb{R}$ = real numbers

Description of Sets

1. Describe the properties of elements of the set
$A = \{x \mid x \in N \text{ and } x \leq 5\}$
2. List elements
$A = \{0, 1, 2, 3, 4, 5\}$
3. Use a recursive formula
$A = \{x_i \mid x_i = x_{i-1} + 1,\ i = 1, 2, 3, 4, 5,\ x_0 = 0\}$
$i = 1 \quad x_1 = x_0 + 1 = 0 + 1 = 1 \qquad i = 2 \quad x_2 = x_1 + 1 = 1 + 1 = 2$
$i = 3 \quad x_3 = x_2 + 1 = 2 + 1 = 3 \qquad i = 4 \quad x_4 = x_3 + 1 = 3 + 1 = 4$
$i = 5 \quad x_5 = x_4 + 1 = 4 + 1 = 5$

* <u>Def</u> Set A is a subset of set B, $A \subseteq B$
if every element in A is an element in B.
Set A is a proper subset of B, $A \subset B$
if $A \subseteq B$ and $A \neq B$

Venn Diagram




Weakness:
cannot distinguish
$A \subseteq B$ from $A \subset B$

elementwise technique of proof

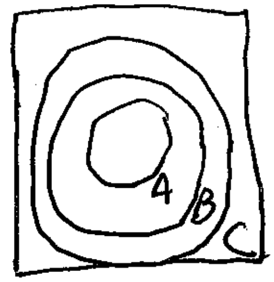
# Properties of Sets

Prove for all sets $A, B, C$:

(Use "elementwise approach"
To prove $\square_1 \subseteq \square_2$, Take $x \in \square_1$ prove $x \in \square_2$)

1. $A \subseteq A$

   $x \in A \Rightarrow x \in A$




2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

   $x \in A \underset{A \subseteq B}{\overset{\text{Given}}{\Longrightarrow}} x \in B \underset{B \subseteq C}{\overset{\text{Given}}{\Longrightarrow}} x \in C$

3. If $A \subseteq B$ and $B \subset C$ then $A \subset C$
   (Same Venn Diagram as 2)

   a) Prove $A \subseteq C$: $x \in A \underset{A \subseteq B}{\overset{\text{Given}}{\Longrightarrow}} x \in B \underset{B \subset C}{\overset{\text{Given}}{\Longrightarrow}} x \in C$

   b) Prove $A \neq C$
   (Find a $y \in C$, with $y \notin A$)
   Since $B \subset C$, there is a $y \in C$, $y \notin B$
   Since $A \subseteq B$, $y \notin A$

4. If $A \subseteq B$ and $A \not\subseteq C$ then $B \not\subseteq C$
   (Find a $y \in B$, with $y \notin C$)
   Since $A \not\subseteq C$ there is a $y \in A$ with $y \notin C$
   Since $A \subseteq B$ $y \in B$

**Def** the universal set, $\mathcal{U}$, is the set of all objects under consideration.

**Def** the set of no elements is called the empty set or null set and is denoted by $\emptyset$ or $\{\ \}$

*Note $\{\emptyset\} \neq \emptyset$, since $\emptyset \in \{\emptyset\}$ and is nonempty

Thm. $\emptyset \subseteq A$ for all sets $A$.

Proof by contradiction

1. Assume opposite (negation) of desired conclusion.

   Assume $\emptyset \not\subseteq A$ for some set $A$.

2. Get a contradiction:

   there is a $y \in \emptyset$, $y \notin A$

   Contradiction

3. Therefore the assumption is false

Def. If $S$ is a set then $|S|$ is the cardinality of $S$ and is the number of elements in the set.

$A = \{a, b, \ldots, z\}$ $|A| = 26$ $|\emptyset| = 0$

* Def. Two sets are equal, $A = B$ iff (if and only if) $A \subseteq B$ and $B \subseteq A$

Note: $\{1,3,5\} = \{5,3,1\} = \{1,1,3,5,5,5\}$

order not important, only membership.

Def. Given a set $S$, the power set of $S$, denoted by $P(S)$ is the set of all subsets of $S$

$P(S) = \{A \mid A \subseteq S\}$ has sets as elements

EX $S = \{a,b,c\}$

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$P(\emptyset) = \{\emptyset\}$

$P(\{a\}) = \{\emptyset, \{a\}\}$

| $\lvert A \rvert$ | $\lvert P(A) \rvert$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 3 | 8 |

Guess $|P(A)| = 2^{|A|}$

# Cartesian Products

<u>Def</u> The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is an ordered collection of elements where $a_1$ is the first element, $a_2$ is the second element, and $a_n$ is the $n^{th}$ element.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \text{ iff } a_1 = b_1, a_2 = b_2, \ldots a_n = b_n$$

Ordered 2 tuple = ordered pair

<u>Def</u> Let $A$ and $B$ be sets. The cartesian product of $A$ and $B$ denoted by $A \times B$ is the set of ordered pairs $(a, b)$ where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

<u>Ex.</u> Let $A = \{1, 2\}$ $B = \{a, b, c\}$

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

$$B \times A = \{(a,1), (b,1), (c,1), (a,2), (b,2), (c,2)\}$$

Note: $|A| = 2$ $|B| = 3$ $|A \times B| = 2 \times 3 = 6$

In general $|A \times B| = |A| \cdot |B|$ for finite sets $A, B$

$A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$, or $|A| = |B|$

<u>Def</u> the cartesian product of the sets $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered n-tuples:

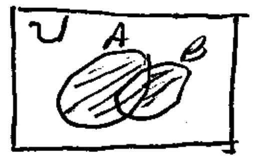
$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2 \ldots a_n \in A_n\}$$

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = A^n$$

# Set Operations

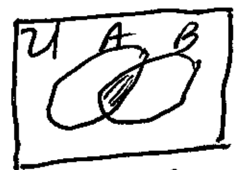
**Def.** the union of two sets, $A$ and $B$, denoted $A \cup B$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

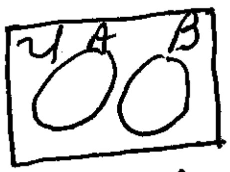



**Def** the intersection of two sets, $A$ and $B$, denoted $A \cap B$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$




**Def** Two sets are disjoint if $A \cap B = \emptyset$




**Thm.** Principle of inclusion-exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

**Def.** the difference of $A$ and $B$, denoted $A - B$

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$


**Def** the complement of $A$, denoted $\overline{A}$

$\overline{A} = \{x \mid x \in U \text{ and } x \notin A\} = \{x \mid x \notin A\}$


* **Thm.** For all sets $A$ and $B$, $A - B = A \cap \overline{B}$

$x \in A - B \overset{\text{Def} -}{\iff} x \in A \text{ and } x \notin B \overset{\text{Def comp}}{\iff} x \in A \text{ and } x \in \overline{B} \overset{\text{Def} \cap}{\iff} x \in A \cap \overline{B}$

$\Rightarrow A - B \subseteq A \cap \overline{B}$

$\Leftarrow A \cap \overline{B} \subseteq A - B$

**Thm.** For all sets $A$, $\overline{\overline{A}} = A$

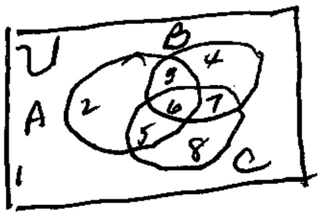
$$x \in \overline{\overline{A}} \overset{\text{Def } \overline{A}}{\iff} x \notin \overline{A} \overset{\text{Def } \overline{A}}{\iff} x \in A \qquad \begin{array}{l} \Rightarrow \overline{\overline{A}} \subseteq A \\ \Leftarrow A \subseteq \overline{\overline{A}} \end{array}$$

**thm.** Set Identities For all sets $A, B, C$

| | Union | Intersection |
|---|---|---|
| Idempotent | $A \cup A = A$ | $A \cap A = A$ |
| Commutative | $A \cup B = B \cup A$ | $A \cap B = B \cap A$ |
| Associative | $A \cup (B \cup C) = (A \cup B) \cup C$ | $A \cap (B \cap C) = (A \cap B) \cap C$ |

Question: Does $A \cup (B \cap C) = (A \cup B) \cap C$

Test with a Venn Diagram




$A \cup (B \cap C) = \{2, 3, 5, 6, 7\}$ not equal

$(A \cup B) \cap C = \{5, 6, 7\}$

Provides a counterexample.

**thm.** Distributive Laws For all sets $A, B, C$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

proof

$x \in A \cap (B \cup C) \overset{\text{Def } \cap}{\iff} x \in A$ and $x \in B \cup C$

$\overset{\text{Def } \cup}{\iff} x \in A$ and $(x \in B$ or $x \in C)$

$\overset{\text{language}}{\iff} (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

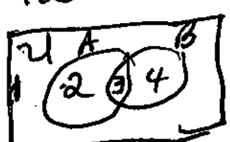
$\overset{\text{Def } \cup, \cap}{\iff} x \in (A \cap B) \cup (A \cap C)$

**thm.** De Morgans Laws. For all sets $A, B$

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Test with Venn Diagram




$\overline{A} = \{1, 4\}$ $\overline{B} = \{1, 2\}$

$\overline{(A \cup B)} = \{1\}$ $\overline{A} \cap \overline{B} = \{1, 2, 4\}$

Proof $x \in \overline{(A \cap B)} \overset{\text{Def } \overline{A}}{\iff} x \notin A \cap B \overset{\text{Def } \cap}{\iff} x \notin (A$ and $B)$

$\overset{\text{*Logic De Morgans}}{\iff} x \notin A$ or $x \notin B \overset{\text{Def } \overline{A}}{\iff} x \in \overline{A}$ or $x \in \overline{B}$

$\overset{\text{Def } \cup}{\iff} x \in \overline{A} \cup \overline{B}$

* Logic not (a and b) $\iff$ not a or not b

not (a or b) $\iff$ not a and not b.

## Lemmas (small theorems)

For all sets $A$:

1. $A \cup \overline{A} = \mathcal{U}$
2. $A \cap \overline{A} = \emptyset$
3. $A \cup \mathcal{U} = \mathcal{U}$
4. $A \cup \emptyset = A$
5. $A \cap \emptyset = \emptyset$

Prove 2 and 5 by contradiction
Prove 1, 3, 4 elementwise

**Lemma** For all sets $A, B$, If $A \subseteq B$ then $A \cap B = A$

$$x \in A \cap B \overset{\text{Def} \cap}{\Longleftrightarrow} x \in A \text{ and } x \in B \overset{\text{Given } A \subseteq B}{\Longleftrightarrow} x \in A$$

* Geometry style approach.

* **Thm** Prove for all sets $A, B, C$

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

(may do elementwise)

$= \overline{A} \cap \overline{(B \cap C)}$ De Morgans

$= \overline{A} \cap (\overline{B} \cup \overline{C})$ De Morgans

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$ Comm prop $\cap$

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$ Comm prop $\cup$

* **Thm** For all sets $A, B$ prove

$$A - (A - B) = A \cap B$$

$= A \cap (\overline{A \cap \overline{B}})$ Thm $A - B = A \cap \overline{B}$

$= A \cap (\overline{A} \cup \overline{\overline{B}})$ De Morgans

$= A \cap (\overline{A} \cup B)$ thm $\overline{\overline{A}} = A$

$= (A \cap \overline{A}) \cup (A \cap B)$ Distributive

$= \emptyset \cup (A \cap B)$ Lemma $A \cap \overline{A} = \emptyset$

$= A \cap B$ Lemma $A \cup \emptyset = A$

**Def** Generalized Unions and Intersections

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x \mid x \in A_1 \text{ and } x \in A_2 \text{ and} \ldots x \in A_n\}$$

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or} \ldots x \in A_n\}$$

**Ex.** Let $\mathcal{U} = P = \{1, 2, \ldots\}$ let $A_i = \{i, i+1, i+2, \ldots\}$ for $i = 1, 2, 3, \ldots$

$A_1 = \{1, 2, 3, \ldots\}$

$A_2 = \{2, 3, \ldots\}$

$A_3 = \{3, 4, 5, \ldots\}$

$$\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \ldots\} = P$$

$$\bigcap_{i=1}^{n} A_i = \{n, n+1, \ldots\} = A_n$$

**Def** Computer Representation of Sets

(Must have a finite Universe) $\mathcal{U} = (a_1, a_2, \ldots, a_n$

Make an arbitrary ordering of

**Def.** A bit (binary digit) is a 1 or 0

A bit string is a finite sequence of bits

Represent a subset $A$ of $\mathcal{U}$ with a bit string of length $n$ where: $i$th bit is 1 if $a_i \in A_i$

$i$th bit is 0 if $a_i \notin A_i$

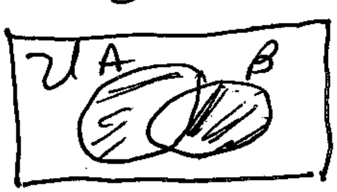
**EX.** $\mathcal{U} = \{1, 2, 3, \ldots, 10\}$

let $A = \{1, 2, 5, 8, 9\}$ $A = 1100100110$

$\overline{A} = 0011011001$

**Def** the symmetric difference of $A$ and $B$, denoted

$A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ and } x \notin (A \text{ and } B)\}$



Thm. For all sets $A, B$,
$A \oplus B = (A \cup B) - (A \cap B)$

Proof. $x \in A \oplus B \xLeftrightarrow{\text{Def } \oplus} x \in A$ or $x \in B$ and $x \notin (A$ and $B)$
$\xLeftrightarrow{\text{Def } \cup, \cap} x \in A \cup B$ and $x \notin (A \cap B)$
$\xLeftrightarrow{\text{Def } A-B} x \in (A \cup B) - (A \cap B)$

Thm. For all sets $A, B$
$A \oplus B = (A - B) \cup (B - A)$

Proof $x \in A \oplus B \xLeftrightarrow{\text{Def } \oplus} x \in A$ or $x \in B$ and $x \notin (A$ and $B)$
$\xLeftrightarrow{\text{language}} (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$
$\xLeftrightarrow{\text{Def } A-B} x \in A - B$ or $x \in B - A$
$\xLeftrightarrow{\text{Def } \cup} x \in (A - B) \cup (B - A)$