Relations and their Properties.

Def Let A and B be sets. A binary relation from A to B is a subset of $A \times B$

If $(a,b) \in R$ we say aRb

If $(a,b) \notin R$ we say $a \not R b$

Ex. Let A = cities in U.S. Let B = states in U.S

Define a relation R from A to B by

aRb if a "is a city in the state" b.

(Miami, FL) $\in R$ (Miami, GA) $\notin R$

Def. A relation from A to A is called a relation on A (subset of $A \times A$)

Ex. Consider the relations on $\mathbb{Z}$:

$R_1 = \{(a,b) \mid a \le b\} = \{(-2,5), (2,3), (-2,0), (1,8), \ldots\}$

$R_2 = \{(a,b) \mid a = b \text{ or } a = -b\} = \{(0,0), (1,1), (1,-1), (-1,1), (-1,-1), \ldots\}$

$R_3 = \{(a,b) \mid a+b \le 3\} = \{(1,2), (-8,6), (-6,-4), \ldots\}$

Ex How many relations are there on a set with n elements?

$|A| = n \quad |A \times A| = n^2 \quad |P(A \times A)| = 2^{n^2}$

Def Properties of a relation R on A

① Reflexive $(a,a) \in R$ for all $a \in A$ (aRa)

② Irreflexive $(a,a) \notin R$ for all $a \in A$ $(a \not R a)$

③ Symmetric If $(a,b) \in R$ then $(b,a) \in R$ for all $a, b \in A$ (If aRb then bRa for all $a,b \in A$)

③ $A = \mathbb{Z}$

$R = \{(x,y) \mid x+y \equiv 0 \bmod 3\}$

Notation:
$x \equiv y \bmod p \iff x-y$ is a multiple
a $x-y = pz$ for some $z \in \mathbb{Z}$

$R = \{(0,3), (-1,4), (-2,-4), (1,2), (3,3), \ldots\}$

Not reflexive: since $(1,1) \notin R$

Not irreflexive: since $(3,3) \in R$

Symmetric: If $(a,b) \in R$ then $a+b = 3z$, for some $z \in \mathbb{Z}$,
then $b+a = 3z$ or $(b,a) \in R$ for all $a,b \in \mathbb{Z}$

Not antisymmetric: since $(1,2) \in R$ and $(2,1) \in R$

Not transitive: since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1) \notin R$
or $(2,4) \in R$ and $(4,5) \in R$ but $(2,5) \notin R$

④ $A = \mathbb{Z}$

$R = \{(x,y) \mid x < y\} = \{(-2,5), (3,6), \ldots\}$

Not reflexive: $(1,1) \notin R$

Irreflexive: $(a,a) \notin R$ for all $a \in \mathbb{Z}$ since $a \not< a$

Not symmetric $(1,3) \in R$ but $(3,1) \notin R$

Antisymmetric If $(a,b) \in R$ and $(b,a) \in R$ then
$a < b$ and $b < a$ (impossible)
so property holds. $(F \to q = T)$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

LOGIC TABLE

Transitive: If $(a,b) \in R$ and $(b,c) \in R$
then $a < b$ and $b < c$ so $a < c$
or $(a,c) \in R$ for all $a,b,c \in \mathbb{Z}$

(called the transitive property of inequality)

④ Antisymmetric If $(a,b) \in R$ and $(b,a) \in R$ then $a=b$ for all $a,b \in A$ (If aRb and bRa then $a=b$ for all $a,b \in A$)

⑤ Transitive If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$ (If aRb and bRc then aRc for all $a,b,c \in A$)

Examples

① $A = N = \{0,1,2,\ldots\}$ $R = \{(a,b) \mid a+b \text{ even}\}$
$= \{(0,2), (1,3), (8,4), \ldots\}$

Reflexive $(a,a) \in R$ for all $a \in N$ since $a+a = 2a$ (even)

Not irreflexive $(1,1) \in R$

Symmetric If $(a,b) \in R$ then $a+b = 2z$, for some $z \in \mathbb{Z}$, then $b+a = 2z$ or $(b,a) \in R$ for all $a,b \in \mathbb{Z}$

Not antisymmetric $(2,4) \in R$ and $(4,2) \in R$

Transitive If $(a,b) \in R$ and $(b,c) \in R$ then $a+b$ is even and $b+c$ is even, so either all a,b,c are even or all a,b,c is odd. Either way $a+c$ is even or $(a,c) \in R$ for all $a,b,c \in N$

② $A = \{1,2,3,4\}$
$R = \{(1,1), (1,3), (4,2), (2,4), (2,3), (3,1)\}$

Not reflexive since $(2,2) \notin R$

Not irreflexive since $(1,1) \in R$

Not symmetric since $(2,3) \in R$ but $(3,2) \notin R$

Not transitive since $(4,2) \in R$ and $(2,4) \in R$ but $(4,4) \notin R$
or $(4,2) \in R$ and $(2,3) \in R$ but $(4,3) \notin R$
etc.

(5) $A = \mathbb{Z}$ $R = \{(x,y) \mid x \leq y\}$

$R = \{(-2,5), (1,1), (4,12), \ldots\}$

Reflexive $(a,a) \in R$ for all $a \in \mathbb{Z}$ since $a \leq a$

Not irreflexive $(1,1) \in R$

Not symmetric $(3,6) \in R$ but $(6,3) \notin R$

Antisymmetric If $(a,b) \in R$ and $(b,a) \in R$ then $a \leq b$ and $b \leq a$ so $a = b$ for all $a, b \in \mathbb{Z}$

(Ex)

6a) $A = P = \{1, 2, \ldots\}$ 6b) $A = \mathbb{Z}$

$R = \{(x,y) \mid x$ divides y, $x \mid y$ or $\frac{y}{x} = z$ for $z \in \mathbb{Z}\}$

a) $R = \{(2,4), (3,6), (1,1), \ldots\}$

Reflexive: $(a,a) \in R$ for all $a \in P$ since $\frac{a}{a} = 1$

Not irreflexive $(1,1) \in R$

Not symmetric $(2,4) \in R$, $(4,2) \notin R$

Antisymmetric:

If $(a,b) \in R$ and $(b,a) \in R$ then $\frac{b}{a} = z_1$ and $\frac{a}{b} = z_2$, $z_1, z_2 \in \mathbb{Z}$

So $b = az_1$ and $a = bz_2$ or $b = bz_2z_1$ or $z_1z_2 = 1$ so $z_1 = z_2 = 1$ or $z_1 = z_2 = -1$

But $a, b \in P$ so $z_1 = z_2 = 1$ and $a = b$.

b) $R = \{(2,4), (1,-1), (-5,25), (2,0), \ldots\}$

Not reflexive $\frac{0}{0}$ undefined $(0,0) \notin R$.

Not irreflexive $(-2,-2) \in R$

Not symmetric $(-2,8) \in R$ $(8,-2) \notin R$

Not antisymmetric $(5,-5) \in R$ and $(-5,5) \in R$

Transitive

Transitive If $(a,b) \in R$ and $(b,c) \in R$ then $\frac{b}{a} = z_1$ and $\frac{c}{b} = z_2$ or $b = az_1$ and $c = bz_2$ or $c = az_1z_2$ or $\frac{c}{a} \in \mathbb{Z}$ so $(a,c) \in R$ for all $a, b, c \in P$ and $a, b, c \in \mathbb{Z}$

(7) $A = \{1,2,3\}$
$R = \{(1,1), (2,2), (3,3)\}$
Reflexive, Not irreflexive $(1,1) \in R$,
Symmetric, Antisymmetric, Transitive

(8) $A = \{1,2,3\}$
$R = \varnothing$
Not reflexive $(1,1) \notin R$, Irreflexive, Symmetric
Antisymmetric, Transitive

(9) $A = \varnothing$
$R = \varnothing$

Reflexive: Assume R is not reflexive on A
there is an $a \in A$ with $(a,a) \notin R$
contradiction

Irreflexive Assume R is not irreflexive on A
there is an $a \in A$ with $(a,a) \in R$
contradiction

Symmetric
Antisymmetric
Transitive

(10) A = people in the world
$R = \{(a,b) \mid a \text{ parent } b\}$

Not reflexive: No one is their own parent
Irreflexive: everyone is not their own parent
Not symmetric: If Bob is parent of John, then John is not parent of Bob.
Antisymmetric a cannot be parent of b and b parent of a at same time
$F \to ? = T$
Not transitive If a parent of b and b parent of c then a is grandparent of c
[illegible]

Combining Relations

<u>Ex</u> let $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$ $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

<u>Def</u> Let R be a relation from a set A to a set B

The inverse relation $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

(a relation from set B to set A)

the complementary relation $\overline{R} = \{(a,b) \mid (a,b) \notin R$

(a relation from set A to set B, $\mathcal{U} = A \times B$)

<u>Ex</u> $A = \mathbb{Z}$

$R = \{(a,b) \mid a < b\}$

$R^{-1} = \{(a,b) \mid a > b\}$

$\overline{R} = \{(a,b) \mid a \geq b\}$

<u>Thm.</u> Let R be a relation on A

Prove R is symmetric iff $R = R^{-1}$

a) If R is symmetric then $R = R^{-1}$

$$(x,y) \in R \underset{R \text{ sym}}{\overset{\text{Given}}{\Longleftrightarrow}} (y,x) \in R \overset{\text{Def } R^{-1}}{\Longleftrightarrow} (x,y) \in R^{-1}$$

b) If $R = R^{-1}$ then R is symmetric on A

$$(x,y) \in R \underset{R = R^{-1}}{\overset{\text{Given}}{\Longrightarrow}} (x,y) \in R^{-1} \overset{\text{Def } R^{-1}}{\Longrightarrow} (y,x) \in R$$

Equivalence Relations

Def An equivalence relation on a set A is a relation which is reflexive, symmetric and transitive.

EX1. R is a relation on the set of strings of letters $\{a, b, \ldots, z\}$ such that $w_1 R w_2$ iff $\ell(w_1) = \ell(w_2)$ where $\ell(w)$ is the length of string w.

Reflexive: $w R w$ for all strings w as $\ell(w) = \ell(w)$

Symmetric: If $w_1 R w_2$ then $\ell(w_1) = \ell(w_2)$, so $\ell(w_2) = \ell(w_1)$ and $w_2 R w_1$ for all strings w_1, w_2.

Transitive: If $w_1 R w_2$ and $w_2 R w_3$ then $\ell(w_1) = \ell(w_2) = \ell(w_3)$ so $\ell(w_1) = \ell(w_3)$ or $w_1 R w_3$ for all strings w_1, w_2, w_3

EX2. R is a relation on the set of real numbers such that $x R y$ iff $x - y \in \mathbb{Z}$

e.g. $3.25 R .25$, $-8 R 6$, $\pi R \pi + 1$

Reflexive: $x R x$ since $x - x = 0 \in \mathbb{Z}$ for all real numbers x.

Symmetric: If $x R y$ then $x - y = z$, $z \in \mathbb{Z}$, so $y - x = -z$, $z \in \mathbb{Z}$ so $y R x$ for all real numbers x, y

Transitive: If $x R y$ and $y R s$ then $x - y = z_1$, $y - s = z_2$ for $z_1, z_2 \in \mathbb{Z}$ then $x - s = z_1 + z_2$, $z_1 + z_2 \in \mathbb{Z}$ so $x R s$ for all real numbers x, y, s

EX3. R is a relation on $\mathbb{Z}$

$R = \{(a, b) \mid a \equiv b \bmod 2\}$ $\quad (a - b = 2z, z \in \mathbb{Z})$

Reflexive $(a, a) \in R$ since $a - a = 0 = 2(0)$ for all $a \in \mathbb{Z}$

Symmetric If $(a, b) \in R$ then $a - b = 2z$, $z \in \mathbb{Z}$ then $b - a = 2(-z)$ or $(b, a) \in R$ for all $a, b \in \mathbb{Z}$

Transitive If $(a, b) \in R$ and $(b, c) \in R$ then $a - b = 2z_1$, $z_1 \in \mathbb{Z}$, $b - c = 2z_2$, $z_2 \in \mathbb{Z}$ then $a - c = 2(z_1 + z_2)$ or $(a, c) \in R$ for all $a, b, c \in \mathbb{Z}$

Note: In EX3, can interchange the mod_2 for mod_3, mod_4, ... and will still have an equivalence relation

<u>Def</u>. Let R be an equivalence relation on a set A. The equivalence class of $a \in A$ is the set of all elements that are related to "a," denoted $[a]$.

$$[a] = \{b \mid bRa\} = \{b \mid (b,a) \in R\}$$

Equivalence classes for EX1, EX2, EX3

<u>EX1</u>. $[\Lambda] = \{\Lambda\}$ Λ = empty string

$[a] = \{a, b, c, \ldots z\}$ $|[a]| = 26$

$[aa] = \{aa, ab, \ldots, zz\}$ $|[aa]| = 26^2$

$[aaa] = \{aaa, \ldots, zzz\}$ $|[aaa]| = 26^3$

$\vdots$

infinite number of equivalence classes

Ex2. $[0] = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\} = \{0+z \mid z \in \mathbb{Z}\}$

$[3.25] = \{\ldots, -1.75, -.75, .25, 1.25, 2.25, 3.25, \ldots\} = \{3.25+z \mid z \in \mathbb{Z}\}$

$[\pi] = \{\ldots, \pi-1, \pi, \pi+1, \pi+2, \ldots\} = \{\pi+z \mid z \in \mathbb{Z}\}$

infinite number of equivalence classes

Ex3. $[0] = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$

$[1] = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$

2 equivalence classes

Let $A = \mathbb{Z}$, $R = \{(a,b) \mid a \equiv b \mod 4\}$ is an equivalence relation

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

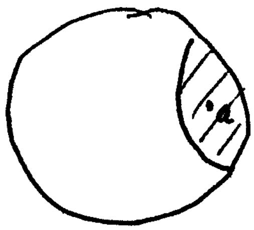
4 equivalence classes.

$\{(a,b) \mid a \equiv b \mod n\}$ will have

Partition

An equivalence relation R on A partitions the set A into equivalence classes. Every element of A is in one and only one equivalence class.

A



$[a] = \{b \mid bRa\}$

Above is proved with the following theorem:

Thm Let R be an equivalence relation on a set A. The following statements are equivalent:

i) aRb

ii) $[a] = [b]$

iii) $[a] \cap [b] \neq \varphi$

Proof.

$i \Rightarrow ii$ If aRb then $[a] = [b]$

Take $x \in [a] \xRightarrow{\text{Def}[\,]} xRa$ (Given aRb) $\xRightarrow{R\text{ trans}} xRb \xRightarrow{\text{Def}[\,]} x \in [b]$

Take $x \in [b] \xRightarrow{\text{Def}[\,]} xRb \xRightarrow[R\text{ sym}]{\text{Given }aRb} xRb, bRa \xRightarrow{R\text{ trans}} xRa \Rightarrow x \in [a]$

So $[a] = [b]$

$ii \Rightarrow iii$ If $[a] = [b]$ then $[a] \cap [b] \neq \varphi$

Since R is reflexive $a \in [a] = [b]$ so $a \in [a] \cap [b] \neq \varphi$

$iii \Rightarrow i$ If $[a] \cap [b] \neq \varphi$ then aRb

Let $x \in [a] \cap [b] \xRightarrow{\text{Def}[\,]} xRa$ and xRb

$\xRightarrow{R\text{ sym}} aRx$ and xRb

$\xRightarrow{R\text{ trans}} aRb$