Mathematical Induction

Technique to prove a statement, $P(n)$, is true for $n=1,2,\ldots$

① Basis Step: $P(1)$ is shown to be true.

② Inductive Step: the implication $P(k) \to P(k+1)$ is shown to be true. for $k=1,2,3,\ldots$

Ex1. Prove $1+2+\cdots+n = \frac{n(n+1)}{2}$ for $n=1,2,\ldots$

Basis Step. $n=1$ $\quad 1 = \frac{1(2)}{2} = 1$

$n=2$ $\quad 1+2 = \frac{2(2+1)}{2} = 3$

Inductive Step.

Assume $1+2+\cdots+k = \frac{k(k+1)}{2}$

* Prove $1+2+\cdots+k+1 = \frac{(k+1)(k+2)}{2}$

Add $k+1$ to both sides of kth case (assumption)

$$1+2+\cdots+k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \; *$$

Ex2. Prove $1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n=1,2,\ldots$

Basis Step $n=1$. $\quad 1^3 = \left(\frac{1(2)}{2}\right)^2 = 1$

$n=2$ $\quad 1^3+2^3 = \left(\frac{2(3)}{2}\right)^2 = 9$

Inductive Step.

Assume $1^3+2^3+\cdots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$

Prove $1^3+2^3+\cdots+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

Add $(k+1)^3$ to both sides of the k^{th} case.

$$1^3+2^3+\cdots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}\left[k^2+4(k+1)\right]$$

$$= \frac{(k+1)^2}{4}(k^2+4k+4) = \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Ex 3. Prove $n < 2^n$ for $n = 1, 2, \ldots$

Basis Step $n=1$ $\quad 1 < 2^1$

Inductive Step.

Assume $k < 2^k$

Prove $(k+1) < 2^{k+1}$

$$k+1 < 2^k + 1 < 2^k + 2^k = 2(2^k) = 2^{(k+1)}$$

Ex 4. Prove $n^3 - n$ is divisible by 3 for $n = 0, 1, 2, \ldots$

Basis Step $n=0$ $\quad 0^3 - 0 = 0 = 3(0)$

Inductive Step.

Assume $k^3 - k$ is divisible by 3

Prove $(k+1)^3 - (k+1)$ is divisible by 3

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= \underbrace{k^3 - k}_{\text{divisibly by 3 by assumption}} + \underbrace{3(k^2 + k)}_{\text{factor of 3 so divisible by 3}}$$

Ex 5. Prove $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for $n = 0, 1, 2, \ldots$

Basis Step $n=0$ $\quad 1 = 2^1 - 1 = 1$

Inductive Step.

Assume $1 + 2 + \cdots + 2^k = 2^{k+1} - 1$

Prove $1 + 2 + \cdots + 2^{k+1} = 2^{k+2} - 1$

Add 2^{k+1} to both sides of k^{th} case

$$1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^1(2^{k+1}) - 1$$

$$= 2^{k+2} - 1$$

Ex 6. Sums from a Geometric Progression

$$a + ar + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1} \text{ for } n = 0, 1, 2, \ldots$$

a = first term

r = fixed ratio, $r \neq 1$

Basis step $n=0$ $\quad a = \dfrac{ar - a}{r-1} = \dfrac{a\cancel{(r-1)}}{\cancel{r-1}}$

Inductive Step

Assume $a + ar + \cdots + ar^k = \dfrac{ar^{k+1} - a}{r-1}$

Prove $a + ar + \cdots + ar^{k+1} = \dfrac{ar^{k+2} - a}{r-1}$

Add ar^{k+1} to both sides of k^{th} case

$$a + ar + \cdots + ar^{k} + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

(EX7) Prove a finite set S with n elements has 2^n subsets. for $n = 0, 1, 2, \ldots$

Basis Step. $n=0$ A set with 0 elements has $2^0 = 1$ subset. $\varnothing \subseteq \varnothing$

Inductive Step.

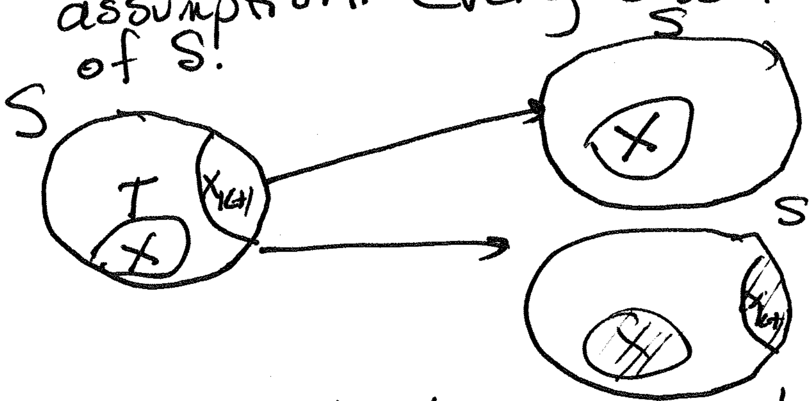
Assume a set with k elements has 2^k subsets

Prove a set with $k+1$ elements has 2^{k+1} subsets

Let $S = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$ $|S| = k+1$

Let $T = S - \{x_{k+1}\}$ $|T| = k$

Since T has k elements, it has 2^k subsets by assumption. Every subset of T is also a subset of S.



$X \subseteq T \subseteq S$ (2^k of these)

$X \cup \{x_{k+1}\} \subseteq S$ (2^k of these)

there are $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets of S.

Ex 8) Prove $2^n < n!$ for $n = 4, 5, 6, \ldots$

Basis Step $n=4$ $\quad 2^4 < 4! = 24$ $\qquad n! = n(n-1)\cdots 1$

$16 < 24$ $\qquad 0! = 1$

Inductive Step

Assume $2^k < k!$

Prove $2^{k+1} < (k+1)!$ $\quad (k = 4, 5, 6, \ldots)$

$$2^{k+1} = 2 \cdot 2^k < 2(k!) \overset{2 < k+1}{<} (k+1) \cdot k! = (k+1)!$$

Ex 9) Prove the generalization of De Morgans Laws:

$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$ for $n = 2, 3, \ldots$

Basis Step $n=2$. $\quad \overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

Inductive Step.

Assume $\overline{A_1 \cap A_2 \cap \cdots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}$

Prove $\overline{A_1 \cap A_2 \cap \cdots \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_{k+1}}$

$\overline{A_1 \cap A_2 \cap \cdots \cap A_{k+1}}$

$= \overline{(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}}$ Associative property of intersection

$= \overline{A_1 \cap A_2 \cap \cdots \cap A_k} \cup \overline{A_{k+1}}$ De Morgans

$= (\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}) \cup \overline{A_{k+1}}$ Assumption

$= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$ Associative property of Union

Ex 10) Prove $6^{n+2} + 7^{2n+1}$ is divisible by 43 for $n = 0, 1, \ldots$

Basis Step $n=0$ $\quad 6^2 + 7^1 = 43 = 43(1)$

Inductive Step.

Assume $6^{k+2} + 7^{2k+1}$ is divisible by 43

Prove $6^{k+3} + 7^{2k+3}$ is divisible by 43 *

$$6^{k+3} + 7^{2k+3} = \underbrace{6(6^{k+2} + 7^{2k+1})}_{\text{divisible by 43 by assumption}} + 7^{2k+1}\underbrace{(-6 + 7^2)}_{43}$$

factor of 43

*Trick to find last term:

$$6^{k+2} + 7^{2k+3} = 6(6^{k+2} + 7^{2k+1}) + X$$

$$6^{k+3} + 7^{2k+3} = 6^{k+3} + 6(7)^{2k+1} + X$$

$$7^{2k+3} - 6(7^{2k+1}) = X$$

$$7^{2k+1}(7^2 - 6)$$

Ex 11 Post office prints 5¢ and 9¢ stamps.
Prove any postage can be made $n = 35¢, 36¢, \ldots$

Basis Step $n = 35¢$ Use 7-5¢ = 35¢

Inductive Step:
Assume k¢ of postage can be made with 5¢ and 9¢ stamps
Prove (k+1)¢ of postage can be made with 5¢ and 9¢ stamps

Two possibilities with the k¢ of postage:

1) there is a 9¢ stamp.
replace it with 2-5¢ stamps
overall postage increments by 1¢ ⇒ (k+1)¢

2) no 9¢ stamps. So all 5¢ stamps.
there are at least 7-5¢ stamps
replace them with 4-9¢ stamps
overall postage increments by 1¢ ⇒ (k+1)¢

Recursively Defined Functions

Def A recursive definition of a function with domain $N = \{0,1,2,..\}$ is given by

i) specifying the value at 0

ii) giving a rule for finding its value at an integer from its values at smaller integers.

Ex. $f(0) = 3$

$f(n+1) = 2f(n) + 3$ for $n = 0,1,2,...$

$n=0$ $f(1) = 2f(0)+3 = 2(3)+3 = 9$

$n=1$ $f(2) = 2f(1)+3 = 2(9)+3 = 21$

$n=2$ $f(3) = 2f(2)+3 = 2(21)+3 = 45$

$n=3$ $f(4) = 2f(3)+3 = 2(45)+3 = 93$

Ex. Give a recursive definition of the factorial function $F(n) = n!$ for $n = 0,1,2,..$

$F(0) = 1$

$F(n+1) = (n+1)F(n)$ for $n = 0,1,2,...$

Ex. the Fibonacci numbers, $f_0, f_1, f_2, ...$ are defined by:

$f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

$n=2$ $f_2 = f_1 + f_0 = 1+0 = 1$ $n=3$ $f_3 = f_2 + f_1 = 1+1 = 2$

$n=4$ $f_4 = f_3 + f_2 = 2+1 = 3$

thm Second Principle of Mathematical Induction

(Always need this theorem when proving by induction with recursively defind functions like Fibonacci)

Basis Cases $p(0), p(1), .. p(q)$ are shown to be true

Inductive Step: the implication $(p(0) \wedge p(1) \wedge .. \wedge p(k) \Rightarrow p(k+1))$ is shown to be true.

Second Principle of Mathematical Induction
Examples:

Ex1 For $f_0=0$, $f_1=1$, $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$

Prove $f_n>\alpha^{n-2}$ for $n=3,4,5,\ldots$ $\alpha=\frac{1+\sqrt{5}}{2}$

Basis cases $n=3$ $f_3>\alpha^1$ $f_3=2>\frac{1+\sqrt{5}}{2}$

$n=4$ $f_4>\alpha^2$ $f_4=3>\left(\frac{1+\sqrt{5}}{2}\right)^2$

Inductive Step

Assume $f_i>\alpha^{i-2}$ for $i=3,4,\ldots k$

Prove $f_{k+1}>\alpha^{k-1}$ * see proof below

$$f_{k+1}=f_k+f_{k-1}>\alpha^{k-2}+\alpha^{k-3}=\alpha^{k-1}$$

* [$x^2-x-1=0$

$x=\frac{1\pm\sqrt{1-4(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$; so $\alpha^2-\alpha-1=0$ or $\alpha^2=\alpha+1$

$\alpha^{k-1}=\alpha^{k-3}\cdot\alpha^2=\alpha^{k-3}(\alpha+1)=\alpha^{k-2}+\alpha^{k-3}$]

Ex2 Recursively define $a_0=1$, $a_1=3$, $a_n=2a_{n-1}-a_{n-2}$ for $n\geq 2$

a) Find $a_2=2a_1-a_0=2(3)-1=5$

$a_3=2a_2-a_1=2(5)-3=7$

$a_4=2a_3-a_2=2(7)-5=9$

b) Guess an explicit formula for a_n:

$a_{n_{exp}}=2n+1$ $n=0,1,2,\ldots$

c. Prove your guess: Prove $a_{n_{rec}}=a_{n_{exp}}$ for $n=0,1,2,\ldots$

Basis cases: $a_{0_{rec}}=1=a_{0_{exp}}=2(0)+1$

$a_{1_{rec}}=3=a_{1_{exp}}=2(1)+1$

Inductive Step:

Assume $a_{i_{rec}}=a_{i_{exp}}=2i+1$ $i=0,\ldots k$

Prove $a_{k+1_{rec}}=a_{k+1_{exp}}=2(k+1)+1=2k+3$

where we need second principle

$$a_{k+1\,rec} = 2a_{k\,rec} - a_{k-1\,rec} = 2a_{k\,exp} - a_{k-1\,exp}$$
$$= 2(2k+1) - (2(k-1)+1) = 4k+2-2k+1 = 2k+3$$

EX3 Recursively define $b_0 = b_1 = b_2 = 1$, $b_n = b_{n-1} + b_{n-3}$ for $n \ge 3$.

a) Calculate $b_3 = b_2 + b_0 = 1+1 = 2$
$b_4 = b_3 + b_1 = 2+1 = 3$
$b_5 = b_4 + b_2 = 3+1 = 4$
$b_6 = b_5 + b_3 = 4+2 = 6$

b) Prove $b_n \ge 2b_{n-2}$ for $n = 3, 4, 5, \ldots$

Basis cases
$b_3 \ge 2b_1 \quad 2 \ge 2(1)$
$b_4 \ge 2b_2 \quad 3 \ge 2(1)$
$b_5 \ge 2b_3 \quad 4 \ge 2(2)$

Inductive Step.
Assume $b_i \ge 2b_{i-2}$ for $i = 3, 4, \ldots k$ second principle
Prove $b_{k+1} = b_k + b_{k-2} \ge 2b_{k-2} + 2b_{k-4}$
$= 2(b_{k-2} + b_{k-4})$
$= 2b_{k-1}$

Extra Credit Only

Recursively Defined Sets

1. Basis Step: an initial collection of elements is specified.
2. Recursive Step - rules for forming new elements in the set from those already known to be in the set.

Def Consider the subset S of the set of integers:

Basis Step $3 \in S$

Recursive Step If $x \in S$, $y \in S$, then $x+y \in S$

$3+3=6$, $6+3=9$, $6+6=12$, ...

(S is positive multiples of 3)

Def. Set Σ^* of strings over an alphabet Σ

Basis Step $\lambda \in \Sigma^*$

Recursive Step. If $w \in \Sigma^*$, $x \in \Sigma$ then $wx \in \Sigma^*$

Ex. If $\Sigma = \{0, 1\}$ $\Sigma^* = \{\lambda, 0, 1, 00, 01, 10, 11, \ldots\}$

If $\Sigma = \{a, b, \ldots, z\}$ $\Sigma^* = \{\lambda, a, \ldots, z, aa, ab, \ldots zz, \ldots\}$

Def Strings combined via concatenation

Basis Step If $w \in \Sigma^*$, then $w\lambda = w$

Recursive Step If $w_1 \in \Sigma^*$, $w_2 \in \Sigma^*$, $x \in \Sigma$ then $w_1 \cdot (w_2 \cdot x) = (w_1 \cdot w_2) \cdot x$

Ex. Recursive definition of length of string w

$l(\lambda) = 0$

$l(wx) = l(w) + 1$ if $w \in \Sigma^*$, $x \in \Sigma$

Def Structural Induction - method to prove results about recursively defined sets.

Basis Step: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.

Recursive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for the new element.

EX Use structural induction to prove that $\ell(xy) = \ell(x) + \ell(y)$ where x and y in Σ^*

(Note: $\ell(\lambda) = 0$, $\ell(wx) = \ell(w) + 1$, $w \in \Sigma^*, x \in \Sigma$)

Basis Step $P(\lambda)$ is true.

$$\ell(x\lambda) = \ell(x) = \ell(x) + 0 = \ell(x) + \ell(\lambda)$$

Recursive Step: $x, y \in \Sigma^*$, $a \in \Sigma$

$$\ell(x(ya)) = \ell(xy) + 1 = \ell(x) + \ell(y) + 1 = \ell(x) + \ell(ya)$$

Above definition Note:

assumption