Functions.

Def. Let A and B be two nonempty sets. A function, denoted by f, from A to B is an assignment of a unique element of B to each element of A.

We write $f(a)=b$ and $f: A \to B$

$(a,b) \in f$ ← Function is a special kind of relation

Formula $f(x)=x+1$ assumes $A=B=\mathbb{R}$ (reals)

Def Let f be a function from A to B

A is called the domain of f

B is called the codomain of f

If $f(a)=b$, b is the image of a; a is the preimage of b.

The range of f is the set of all images of elements of the domain, A.

f "maps" A to B

Ex The following are relations from A to B. Which ones are functions from A to B?

$A=\{a,b,c\}$ $B=\{1,2,3\}$

R_1:	R_2:	R_3:	R_4:
$a \to 1$	$a \to 1$	$a \to 1$	$a \to 1$
$a \to 2$	$b \to 2$	$b \to 1$	$b \to 2$
$b \to 3$		$c \to 1$	$c \to 3$
$c \to 3$			
Not a function	Not a function	function	function.

Def. Let f be a function from A to B. Let $S \subseteq A$. The image of S, denoted $f(S)$ is the subset of B which consists of the images of the elements of S $f(S)=\{f(s) \mid s \in S\}$

Ex. Let $A = \{a,b,c,d,e\}$ $B = \{1,2,3,4\}$

f

$a \to 2$
$b \to 1$
$c \to 4$
$d \to 1$
$e \to 1$

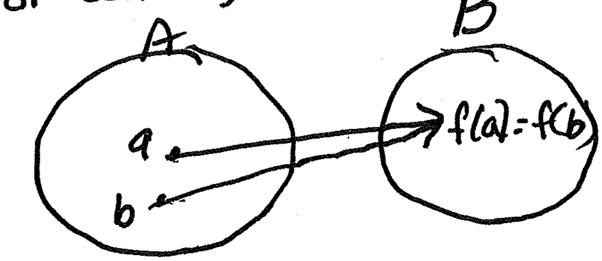
Let $S = \{b,c,d\}$ $f(S) = \{1,4\}$

$A = \{a,b,c,d,e\}$ $f(A) = \{1,2,4\}$
(range)

Properties of a function $f: A \to B$

Def. f is one-to-one, 1:1, or injective if $a \neq b$ implies $f(a) \neq f(b)$ for all $a, b \in A$.

Counterexample:



Ex. $A = \{a,b,c,d\}$ $B = \{1,2,3,4\}$

f_1

$a \to 1$
$b \to 1$
$c \to 2$
$d \to 3$

Not 1:1
$f(a) = f(b) = 1$

f_2

$a \to 1$
$b \to 2$
$c \to 3$
$d \to 4$

Is 1:1

Def. f is onto or surjective if $f(A) = B$

Note: $f(A) \subseteq B$ for all functions $f: A \to B$

To prove f is onto, prove $B \subseteq f(A)$:

Take $b \in B$, find an $a \in A$ with $b = f(a)$

Ex. $A = \{a, b, c, d\}$ $B = \{1, 2, 3\}$

f_1

$a \to 1$
$b \to 2$
$c \to 3$
$d \to 1$

f is onto
$f(A) = B$

f_2

$a \to 1$
$b \to 2$
$c \to 1$
$d \to 2$

f is not onto
$3 \notin f(A)$

Ex. 1. $f: N \to N$ $f(x) = x^2$

1:1 $n_1 \neq n_2 \Rightarrow n_1^2 \neq n_2^2$ $(n_1, n_2 \in N) \Rightarrow f(n_1) \neq f(n_2)$

Not onto $2 \notin f(N)$ etc.

2. $f: Z \to Z$ $f(x) = x^2$

Not 1:1 $f(-1) = f(1) = 1$

Not onto $-3 \notin f(Z)$

3. $f: N \to N$ $f(x) = x + 1$

1:1 $n_1 \neq n_2 \Rightarrow n_1 + 1 \neq n_2 + 1 \Rightarrow f(n_1) \neq f(n_2)$ $n_1, n_2 \in N$

Not onto $0 \notin f(N)$

4. $f: Z \to Z$ $f(x) = x + 1$

1:1 $z_1 \neq z_2 \Rightarrow z_1 + 1 \neq z_2 + 1 \Rightarrow f(z_1) \neq f(z_2)$ $z_1, z_2 \in Z$

Onto For $z \in Z$, $z - 1 \in Z$, $f(z-1) = z - 1 + 1 = z$

5. $f: N \to N_e$ $(N_e = \{0, 2, 4, 6, \ldots\})$ $f(x) = 2x$

1:1 $n_1 \neq n_2 \Rightarrow 2n_1 \neq 2n_2 \Rightarrow f(n_1) \neq f(n_2)$ $n_1, n_2 \in N$

Onto $n_e \in N_e$ $n_e = f\left(\frac{n_e}{2}\right)$ $\left(\frac{n_e}{2} \in N\right.$ since n_e even$\left.\right)$

6. $f: N \times N \to N$ $f(x, y) = x \cdot y$

Not 1:1 $f(2,3) = f(3,2) = f(6,1) = 6$

Onto $n \in N$, $(n,1) \in N \times N$, $f(n,1) = n \cdot 1 = n$

Def. A function f is called a bijection, or a one-to-one correspondence if it is both one-to-one and onto.

Def. Let A be a set. The identity function on A is the function $i_A : A \to A$, $i_A(x) = x$.

Thm. The identity function, $i_A : A \to A$, $i_A(x) = x$ is a bijection:

Proof. 1:1 $\quad x_1 \neq x_2 \Longrightarrow i_A(x_1) \neq i_A(x_2)$ for $x_1, x_2 \in A$

Onto $\quad$ Take $x \in A$ $\quad x = i_A(x)$

If a function is not both 1:1 and onto, there is no inverse function:

EX $A = \{1,2,3\}$ $\quad B = \{a,b\}$

f:
$1 \to a$
$2 \to b$
$3 \to a$

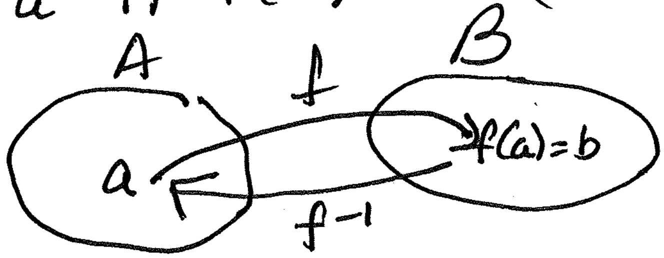
$a \to 1$
$b \to 2$
$a \to 3$
Not a function $A \to B$

$A = \{1,2\}$ $\quad B = \{a,b,c\}$

f:
$1 \to a$
$2 \to b$

$a \to 1$
$b \to 2$
Not a function $B \to A$

Def. Let f be a bijection from A to B. The inverse function, $f^{-1} : B \to A$ is defined by $f^{-1}(b) = a$ if $f(a) = b$ (f is called invertible)



Ex. $A = \{1,2,3\}$ $B = \{a,b,c\}$

$$\begin{array}{ccc} f & \quad & f^{-1} \\ 1 \to a & & a \to 1 \\ 2 \to b & & b \to 2 \\ 3 \to c & & c \to 3 \end{array}$$

Ex. $f: N \to N_e$ $f(x) = 2x$ is a bijection

To find the inverse function, f^{-1},

① Let $y = f(x)$ — $y = 2x$

② Switch x & y — $x = 2y$

③ Solve for y — $y = \frac{x}{2}$

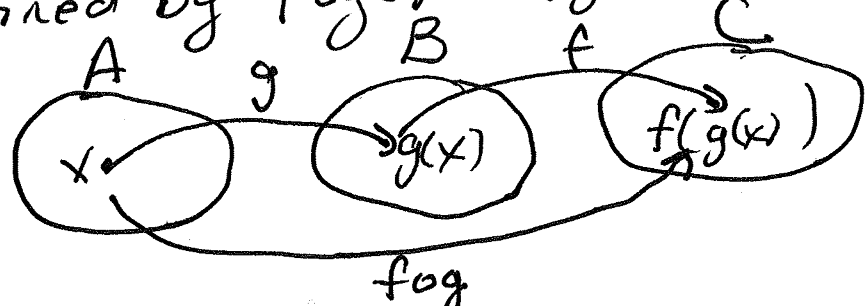
④ $y = f^{-1}(x)$ — $f^{-1}(x) = \frac{x}{2}$

⑤ Switch domain and codomain — $f^{-1}: N_e \to N$

Def. Let $g: A \to B$, $f: B \to C$ be functions

the composition of f with g, $f \circ g: A \to C$

is defined by $f \circ g(x) = f(g(x))$



Ex $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 2x+3$; $g: \mathbb{R} \to \mathbb{R}$ $g(x) = x^2$

$f \circ g(x) = f(g(x)) = f(x^2) = 2x^2 + 3$

$g \circ f(x) = g(f(x)) = g(2x+3) = (2x+3)^2 = 4x^2 + 12x + 9$

Thm Say f is invertible $f: A \to B$ then

$f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$

Proof $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$ $\quad f^{-1} \circ f = i_A$ $\quad a \in A$

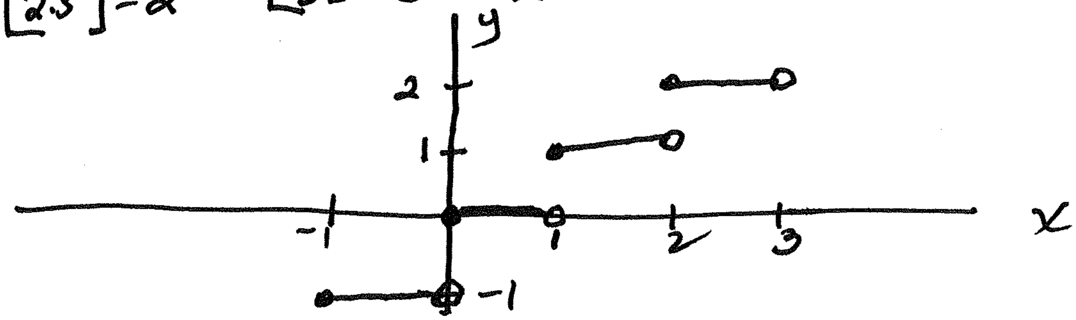
$f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$ $\quad f \circ f^{-1} = i_B$ $\quad b \in B$

Ex. $f: \mathbb{R} \to \mathbb{Z}$ $f(x) = \lfloor x \rfloor = [x]$

floor function; greatest integer function

(defined as the greatest integer less than or equal to x)

$[2.5] = 2$ $[3] = 3$ $[-3] = -3$ $[-3.5] = -4$



Def. A sequence is a function whose domain is $N = \{0, 1, 2, \ldots\}$ or $P = \{1, 2, 3, \ldots\}$

$a_n = f(n)$ (n^{th} term of sequence)

Ex $f: P \to Q$ $a_n = \frac{1}{n}$ $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

$f: N \to \mathbb{Z}$ $b_n = (-1)^n$ $1, -1, 1, -1, 1, \ldots$

$f: N \to \mathbb{Z}$ $c_n = 5^n$ $1, 5, 25, 125, \ldots$

Def. Summation Notation

$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$ $a_j = f(j)$

$m, n \in \mathbb{Z}$, $m \le n$

n = upper limit m = lower limit

Ex $\sum_{j=1}^{5} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ $\sum_{j=1}^{5} 6 = 6+6+6+6+6$

Def. Product Notation

$\prod_{j=m}^{n} a_j = (a_m)(a_{m+1}) \cdots (a_n)$ $m, n \in \mathbb{Z}$ $m \le n$

Ex $\prod_{j=1}^{5} j^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2$