# Homework Assignments

## Set Theory

p. 125 1-11 odd, 17-27 odd, 31, 31
p. 136 3, 5-11*, 13*, 15a, 16*, 18*, 21-25* odd, 27-31* odd, 37-41 odd, 47-53 odd
*(prove elementwise)
Set Theory Supplement problems
Note: solutions to even numbered problems and solutions p.136 39, 41 included in supplement

## Relations

p. 581 1-17 odd, 27, 31, 35, 49-53 odd
(complete solutions to #7 in supplemental relations)
p. 615 1, 3, 11-15odd, 23, 31, 37-45 odd
Relations Supplement problems (solutions p. 581, 7 and p. 616, 30, 32 included)

## Functions

p. 152 1, 3, 11-15 odd, 23, 31, 37-45 odd
p. 167 1, 3, 9, 11a, 31, 43
Function Supplement problems (solutions p.l54 30, 31 included)

## Cardinality

p. 176 1, 3, 15 17, 27, 29
Cardinality Supplement problems

## Logic

p. 12 1-13 odd, 15 a,b,c 19-37 odd, 43
p. 34 1-33 odd
p. 53 1-19 odd, 23 (Universe – students in class), 25, 33
p. 66 27 a-h, 31 a-c, 33, 37 b,c
Logic Supplement problems

## Induction and Recursion

p. 329 5-15 odd, 21-25 odd, 41-45 odd
p. 357 1-13 odd, 27 a,c 33
Induction and Recursion Supplement problems

## Combinatorics

p. 396 1-15 odd, 25-35 odd
p. 405 1, 3, 9, 31-35 odd
p. 413 1-27 odd, 31, 33
p. 421 1-9 odd, 23
Combinatorics Supplement problems

Graph Theory

p. 649 3-9
p 665 1, 2,5 ,7,8, 35 a, b 43
p. 675 5-8, 13-15, 19-21, 35-37, 41, 42, 61-63
p.689 1, 3
p. 703 1-6, 13-15
p. 725 5-7
p. 732 3-9
p.755 1
p 783 7, 10, 13, 17 a, b 23, 24
Graph Supplement problems
Note: solutions to even numbered problems on supplement

Boolean Algebra

p.818 5
p.822 1, 3
p 827 1-5
p.841 3, 6a,b, c*, 7, 12
*For 6c, use f(x ,y,z)= $\bar{x}yz + (x+\bar{z})(\bar{y}+\bar{z})$
Boolean algebra Supplement problems
Note: solutions to even numbered problems on supplement

# Set Theory Supplement Problems

*Prove for all set A, B, C*

1. a) $A \cap \overline{A} = \varnothing$ B) $A \cup \overline{A} = U$

2. a) If $A \subseteq B$ then $\overline{B} \subseteq \overline{A}$

   b) If $\overline{B} \subseteq \overline{A}$ then $A \subseteq B$

3. If $A \subseteq B$ then $A - (\overline{B} \cap A) = A$

4. If $A \cap B = \varnothing$ then $A - (\overline{B} \cap A) = \varnothing$

5. $(A \cap B) \times C = (A \times C) \cap (B \times C)$

6. $(A \cup B) \times C = (A \times C) \cup (B \times C)$

*Solutions*

1. a) Assume $A \cap \overline{A} \neq \varnothing$ Then there is a $y \in A \cap \overline{A} \Rightarrow y \in A$ and $y \notin A$
   *Contradiction*

   b) $x \in A \cup \overline{A} \xleftrightarrow{Def \overline{A}} x \in A$ or $x \notin A \xleftrightarrow{Def U} x \in U$

2. a) $x \in \overline{B} \longrightarrow x \notin B \xrightarrow{A \subseteq B} x \notin A \longrightarrow x \in \overline{A}$

   b) $x \in A \longrightarrow x \notin \overline{A} \xrightarrow{\overline{B} \subseteq \overline{A}} x \notin \overline{B} \longrightarrow x \in B$

**3.** $A - (\overline{B} \cap A)$

| | |
|---|---|
| $A \cap \overline{(\overline{B} \cap A)}$ | *thm* $A - B = A \cap \overline{B}$ |
| $A \cap (\overline{\overline{B}} \cup \overline{A})$ | *De Morgans* |
| $A \cap (B \cup \overline{A})$ | *thm* $\overline{\overline{A}} = A$ |
| $(A \cap B) \cup (A \cap \overline{A})$ | *Distributive law* |
| $(A \cap B) \cup \varnothing$ | *Lemma* $A \cap \overline{A} = \varnothing$ |
| $A \cap B$ | *Lemma* $A \cup \varnothing = A$ |
| $A$ | *If* $A \subseteq B$ then $A \cap B = A$ |

**4.** $A - (\overline{B} \cap A)$

| | |
|---|---|
| $A \cap \overline{(\overline{B} \cap A)}$ | *thm* $A - B = A \cap \overline{B}$ |
| $A \cap (\overline{\overline{B}} \cup \overline{A})$ | *De Morgans* |
| $A \cap (B \cup \overline{A})$ | *thm* $\overline{\overline{A}} = A$ |
| $(A \cap B) \cup (A \cap \overline{A})$ | *Distributive law* |
| $(A \cap B) \cup \varnothing$ | *Lemma* $A \cap \overline{A} = \varnothing$ |
| $A \cap B$ | *Lemma* $A \cup \varnothing = A$ |
| $\varnothing$ | *Assumption* |

5. $(x,y)\in(A\cap B)\times C \xleftrightarrow{def\times} x\in A\cap B$ and $y\in C \xleftrightarrow{def\cap} x\in A$ and $x\in B$ and $y\in C$ $\xleftrightarrow{Logic}$ $(x\in A$ and $y\in C)$ and $(x\in B$ and $y\in C)$ $\xleftrightarrow{def\times}$ $(x,y)\in A\times C$ and $(x,y)\in B\times C \xleftrightarrow{def\cap} (x,y)\in(A\times C)\cap(B\times C)$

6. $(x,y)\in(A\cup B)\times C \xleftrightarrow{def\times} x\in A\cup B$ and $y\in C \xleftrightarrow{def\cup} x\in A$ or $x\in B$ and $y\in C$ $\xleftrightarrow{Logic}$ $(x\in A$ and $y\in C)$ or $(x\in B$ and $y\in C)$ $\xleftrightarrow{def\times}$ $(x,y)\in A\times C$ or $(x,y)\in B\times C$ $\xleftrightarrow{def\cup} (x,y)\in(A\times C)\cup(B\times C)$

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6. a) $x\in A\cup\varnothing \xleftrightarrow{def\cup} x\in A$ or $x\in\varnothing \longleftrightarrow x\in A$

b) $x\in A\cap U \xleftrightarrow{def\cap} x\in A$ and $x\in U \xleftrightarrow{defUniverse} x\in A$

8. a) $x\in A\cup A \xleftrightarrow{def\cup} x\in A$ or $x\in A \longleftrightarrow x\in A$

b) $x\in A\cap A \xleftrightarrow{def\cap} x\in A$ and $x\in A \longleftrightarrow x\in A$

10. a) $x\in A-\varnothing \xleftrightarrow{def-} x\in A$ and $x\notin\varnothing \longleftrightarrow x\in A$ (all $x\notin\varnothing$)

b) Assume $\varnothing - A\neq\varnothing$ Then there is a $y\in\varnothing-A \rightarrow y\in\varnothing$ and $y\notin A$
*Contradiction*

16. a) $x\in A\cap B \rightarrow x\in A$ and $x\in B \rightarrow x\in A$

b) $x\in A \rightarrow x\in A$ or $x\in B \rightarrow x\in A\cup B$

c) $x\in A-B \rightarrow x\in A$ and $x\notin B \rightarrow x\in A$

d) $A\cap(B-A)$

| | |
|---|---|
| $A\cap(B\cap\overline{A})$ | *thm* $A-B=A\cap\overline{B}$ |
| $A\cap(\overline{A}\cap B)$ | *Commmutative Law* |
| $(A\cap\overline{A})\cap B$ | *Associative Law* |
| $\varnothing\cap B$ | *Lemma* $A\cap\overline{A}=\varnothing$ |
| $\varnothing$ | *Lemma* $A\cap\varnothing=\varnothing$ |

e) $A\cup(B-A)$

| | |
|---|---|
| $A\cup(B\cap\overline{A})$ | *thm* $A-B=A\cap\overline{B}$ |
| $(A\cup B)\cap(A\cup\overline{A})$ | *Distributive Law* |
| $(A\cup B)\cap U$ | *Lemma* $A\cup\overline{A}=U$ |
| $A\cup B$ | *Lemma* $A\cap U=A$ |

**18.** a) $x \in (A \cup B) \xrightarrow{def \cup} x \in A$ or $x \in B \Rightarrow x \in A$ or $x \in B$ or $x \in C \xrightarrow{def \cup}$ $x \in A \cup B \cup C$

b) $x \in A \cap B \cap C \xrightarrow{def \cap} x \in A$ and $x \in B$ and $x \in C \Rightarrow x \in A$ and $x \in B \xrightarrow{def \cap}$ $x \in A \cap B$

c) $x \in (A - B) - C \xrightarrow{def -} x \in A - B$ and $x \notin C \xrightarrow{def -} (x \in A$ and $x \notin B)$ and $x \notin C \Rightarrow x \in A$ and $x \notin C \xrightarrow{def -} x \in A - C$

d) Assume $(A - C) \cap (C - B) \neq \varnothing$. Then there is a $y \in (A - C) \cap (C - B) \xrightarrow{def \cap}$ $y \in A - C$ and $y \in C - B \xrightarrow{def -} (y \in A$ and $y \notin C)$ and $(y \in C$ and $y \notin A) \Rightarrow$ $y \in C$ and $y \notin C$ contradiction.

e) $(B - A) \cup (C - A)$

| | |
|---|---|
| $(B \cap \overline{A}) \cup (C \cap \overline{A})$ | thm $A - B = A \cap \overline{B}$ |
| $(B \cup C) \cap \overline{A}$ | Distributive law |
| $(B \cup C) - A$ | thm $A - B = A \cap \overline{B}$ |

**39.** If $A \oplus B = A$ then $B = \varnothing$

Proof: Assume $B \neq \varnothing$ then there is a $b \in B$
Either $b \in A$ or $b \notin A$

If $b \in A \xrightarrow{*} b \notin A \oplus B = A$ *Contradiction*
If $b \notin A \xrightarrow{*} b \in A \oplus B = A$ *Contradiction*

* *definition of* $\oplus$

**41.** Proof by contradiction. Assumee $A \not\subseteq B$
Then there is a $y \in A$ and $y \notin B$.

say $y \in C \xrightarrow{\oplus} y \in B \oplus C \xrightarrow{*} y \in A \oplus C$ *Contradiction* $(y \in A$ and $y \in C)$

say $y \notin C \xrightarrow{\oplus} y \in A \oplus C \xrightarrow{*} y \in B \oplus C$ *Contradiction* $(y \notin B$ and $y \notin C)$

**Assumption*

This proves $A \subseteq B$
To prove $B \subseteq A$, interchange $A$'s & $B$'s in proof

**Relations supplementary problems**

1. Let $R_1$ and $R_2$ be relations on A
   Prove $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

2. If $R_1$ is transitive on A and $R_2$ is transitive on A, must $R_1 \cup R_2$ be transitive on A? must $R_1 \cap R_2$ be transitive on A?

3. Prove if $R_1$ is symmetric on A and $R_2$ is symmetric on A then $R_1 \cup R_2$ and $R_1 \cap R_2$ are symmetric on A.

4. Prove if R is antisymmetric on A then $R \cap R^{-1} \subseteq D$ where $D = \{(a,a) \mid a \in A\}$

5. Which of the following describe equivalence relations. Either specifies which properties fail or list the equivalence classes.

   a) A= set of straight lines in the plane :
      $L_1 \; R \; L_2$ if $L_1$ and $L_2$ are perpendicular

   b) A= set of Americans living in 50 states:
      $P_1 \; R \; P_2$ if $P_1$ and $P_2$ live in the same state (only one residency allowed)

   c) $A = \{0,1,2,\ldots\}$
      $R = \{(m,n) \mid m^2 \equiv n^2 \bmod 3\}$

## Solutions relations

1. $(x,y)\in(R_1\cup R_2)^{-1}\xleftrightarrow{defR^{-1}}(y,x)\in R_1\cup R_2\xleftrightarrow{def\cup}(y,x)\in R_1$ or $(y,x)\in R_2$
$\xleftrightarrow{defR^{-1}}(x,y)\in R_1^{-1}$ or $(x,y)\in R_2^{-1}\xleftrightarrow{def\cup}(x,y)\in R_1^{-1}\cup R_2^{-1}$

2. $R_1\cup R_2$ is not necessarily transitive: Counterexample:

$A=\{1,2,3\}$ $\quad R_1=\{(1,1),(1,2),(2,1),(2,2)\}$ is transitive on A
$\quad R_2=\{(2,2),(2,3),(3,2),(3,3)\}$ is transitive on A

but $R_1\cup R_2$ is not transitive on A: $(1,2)\in R_1\cup R_2$ $(2,3)\in R_1\cup R_2$
but $(1,3)\notin R_1\cup R_2$

$R_1\cap R_2$ is transitive on A: Proof
$(x,y)\in R_1\cap R_2$ and $(y,z)\in R_1\cap R_2\xrightarrow{def\cap}(x,y)\in R_1$, and $(y,z)\in R_1$
and $(x,y)\in R_2$ and $(y,z)\in R_2\xrightarrow[(R_1,R_{2trans})]{Ass}(x,z)\in R_1$ and $(x,z)\in R_2$

$\xrightarrow{Def\cap}(x,z)\in R_1\cap R_2$

3. $(x,y)\in R_1\overset{(\cap)}{\cup}R_2\Rightarrow(x,y)\in R_1\overset{(and)}{\text{or}}(x,y)\in R_2$

$\xrightarrow[(R_1,R_{2Sym})]{Assumt*}(y,x)\in R_1\overset{(and)}{\text{or}}(y,x)\in R_2\Rightarrow(y,x)\in R_1\overset{(\cap)}{\cup}R_2$

4. $(x,y)\in R\cap R^{-1}\xrightarrow{Def\cap}(x,y)\in R$ and $(x,y)\in R^{-1}$

$\xrightarrow{DefR^{-1}}(x,y)\in R$ and $(y,x)\in R$

$\xrightarrow[(Rantisym)]{Ass}x=y\xrightarrow{defD}(x,y)\in D$

5. a) Not reflexive, not transitive

b) Each equivalence class consist of Americans living in the same state. (50 classes)

c) Reflexive $(x,x)\in R$ for all $x\in A$ since $x^2-x^2=0=3\times 0$
Symmetric If $(x,y)\in R$ then $x^2-y^2=3k,\ k\in Z\Rightarrow y^2-x^2=3(-k)\Rightarrow(y,x)\in R$

Transitive If $(x,y) \in R$ and $(y,z) \in R$ then

$$x^2 - y^2 = 3k_1,\ y^2 - z^2 = 3k_2,\ k_1, k_2 \in Z \Rightarrow$$
$$x^2 - z^2 = 3(k_1 + k_2) \Rightarrow (x,z) \in R$$

Two equivalence classes; $[0] = \{0,3,6,9...\}, [1] = \{1,2,4,5,7,8,...\}$

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7. a) Not reflexive $1 \not R 1$ since 1=1
   Irreflexive $x \not R x$ since $x = x$ fn all $x \in Z$
   Symmetric If $xRy \Rightarrow x \neq y$ then $y \neq x \Rightarrow yRx$
   Not antisymmetric $1\ R\ 2$ and $2\ R\ 1$
   Not transitive $1\ R\ 2$ and $2\ R\ 1$ but $1 \not R 1$

   b) Not reflexive $0 \not R 0$ since $0.0 \not\geq 1$
   Not irreflexive $1\ R\ 1$ since $1.1 \geq 1$
   Symmetric If $xy \geq 1$ then $yx \geq 1$
   Not antisymmetric $1\ R\ 2$ and $2\ R\ 1$
   Transitive If $xy \geq 1$ and $yz \geq 1$ then either
   x,y,z $\geq 1$ or $x,y,z \leq -1$ so $xz \geq 1$ in either case.

   c) Not reflexive $1 \neq 1+1$ or $1 \neq 1-1$ so $1 \not R 1$
   Irreflexive $x \not R x$ for all $x \neq x+1$ or $x \neq x-1$
   Symmetric If $xRy$ then $x = y+1$ or $x = y-1$ then $y = x-1$ or $y = x+1$ or $y\ R\ x$
   Not antisymmetric $1\ R\ 2$ and $2\ R\ 1$
   Not transitive $1\ R\ 2$ and $2\ R\ 1$ but $1 \not R 1$
   or $1\ R\ 2$ and $2\ R\ 3$ but $1 \not R 3$

   d) Reflexive x $R$ x for all $x \in z$ since $x - x = 7 \times 0$
   Not irreflexive $1\ R\ 1$
   Symmetric if x $R$ y then $x - y = 7z$ for some $z \in Z$
   Then $y - x = 7(-z)$ so y $R$ $x$ for all $x, y, \in Z$
   Not antisymmetric $1\ R\ 8$ $8\ R\ 1$
   Transitive If $x\ R\ y$ and $y\ R\ z$ then $x - y = 7z_1$, and
   $y - z = 7z_2$ for some $z_1, z_2 \in Z$ then
   $x - z = 7(z_1 + z_2)$ or $x\ R\ z$ for all $x, y, z \in Z$

   e) Reflexive $x\ R\ x$ for all $x \in Z$ since $x = x\ (1)$
   Not irreflexive $1\ R\ 1$
   Not symmetric $4\ R\ 2$ $4=2z$ but $2 \not R 4$
   Not antisymmetric $5\ R\ {-5}$ and $-5\ R\ 5$ but $5 \neq -5$

Transitive If $x\ R\ y$ and $y\ R\ z$ then $x = y\ z_1$ and $y = z \times z_2$
For some $z_1, z_2 \in Z$ then $x = (zz_2)z_1 = z(z_2 z_1)$
or $x\ R\ z$ for all $x, y, z \in Z$

f) Reflexive $x\ R\ x$ for all $x \in Z$ ( $x$ either negative or non negative)
Not irreflexive $1\ R\ 1$
Symmetric If $x\ R\ y$, $x$ or $y$ both negative or non negative, so $y\ R\ x$
Not antisymmetric $1\ R\ 2$ and $2\ R\ 1$
Treansitive If $x\ R\ y$ and $y\ R\ z$ then $x, y, z$ all negative,
or all non negative so $x\ R\ z$

g) Not reflexive $2 \neq 2^2 = 4$ so $(2,2) \notin R$
Not irreflexive $(1,1) \in R$ since $1 = 1^2$
Not symmetric $(4,2) \in R$ since $4 = 2^2$ but $(2,4) \notin R$ since $2 \neq 4^2$
Antisymmetric If (x,y) $\in R$ then $x = y^2 \geq y$ and if (y,x) $\in R$
then $y = x^2 \geq x$ so $x \geq y \wedge y \geq x$ then $x = y$
Not transitive $16\ R\ 4\ (16 = 4^2)$, $4\ R\ 2\ (4 = 2^2)$ but $16 \not R 2$ $16 \neq 2^2$

h) Not reflexive $(2,2) \notin R$ $2 \not\geq 2^2$
Not irreflexive $(1,1) \in R$ $1 \geq 1^2$
Not symmetric $(4,2) \in R$ $4 \geq 2^2$ but $2 \not\geq 4^2$ so $(2,4) \notin R$
Antisymmetric If $(x,y) \in R$ then $x \geq y^2 \geq y$
And $(y,x) \in R$ so $y \geq x^2 \geq x$ then $x = y$ for all $x, y \in Z$
Transitive If $(x,y) \in R$ and $(y,z) \in R$
Then $x \geq y^2$ and $y \geq z^2$ (since $y^2 \geq y$)
$x \geq y^2 \geq y \geq z^2$ so $x \geq z^2$ or $(x,z) \in R$ for all $x, y, z \in Z$

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30. [010] = {010,0101,0100,01010,01011,01001,01000,...}
[1011] = {101,1011,1010,...}
[11111] = {111,1110,1111,11100,11101,...}
[01010101] same as [010]

32 a) [010] = {010, 000, 0100, 0101, 0000, 0001, ... }
b) [1011] = {101, 111, 1010, 1011, 1110, 1111, ... }
[11111] = Same as (b)
[01010101] = Same as (a)

## Functions Supplement Problem

For each function below, prove whether the function is one-to-one, onto, neither or a bijection.

| | | |
|---|---|---|
| 1. | $f: R \to Z,$ | $f(x) = [x]$ *(Greater integer less than or equal to x)* |
| 2. | $g: N \to N$ | $g(x) = 2x$ |
| 3. | $h: Q^+ \to Q^+$ | $h(x) = \frac{1}{x}$   $Q^+ =$ Positive Rationals |
| 4. | $g: P \times P \to Q^+$ | $g((a,b)) = \frac{a}{b}$ |
| 5. | $f: Z \times Z \to Z$ | $f((a,b)) = a + b$ |
| 6. | $f: N \to N_0$ | $N_0 = \{1,3,5,7,...\}$   $f(n) = 2n+1$ |
| 7. | $p: N \to \{0,1\}$ | $p(n) = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 & \text{if } n \text{ is (not) prime} \end{cases}$ |
| 8. | $k: R \to R$ | $k(x) = \lvert x \rvert$ |

$P = \{1,2,3,4,...\}$
$N = \{0,1,2,3,...\}$
$Z = \{...,-3,-2,1,0,1,2,3,...\}$
$Q =$ Rationals
$R =$ reals

*Selected Answers*
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40. a) $y \in f(S \cup T) \Leftrightarrow y = f(x) \in f(S \cup T) \Leftrightarrow x \in S \cup T \Leftrightarrow x \in S \text{ or } x \in T$
$\Leftrightarrow f(x) \in f(S) \text{ or } f(x) \in f(T) \Leftrightarrow f(x) \in f(S) \cup f(T)$

b) $y \in f(S \cap T) \Leftrightarrow y = f(x) \in f(S \cap T) \Leftrightarrow x \in S \cap T \Leftrightarrow x \in S \text{ and } x \in T$
$\Rightarrow f(x) \in f(S) \text{ and } f(x) \in f(T) \Leftrightarrow f(x) \in f(S) \cap f(T)$

Arrow doesn't
Always reverse
See # 41 next page

4) Define $f: A \to B$ $A = \{1,2,3\}$ $B = \{a,b\}$

$$\begin{array}{ccc} & f & \\ 1 & \to & a \\ 2 & \to & b \\ 3 & \to & a \end{array}$$

Let $S = \{1,2\}$ $T = \{3\}$

$S \cap T = \varnothing$, $f(S \cap T) = \varnothing$

But $f(S) = \{a,b\}$, $f(T) = \{a\}$.
So $f(S) \cap f(T) = \{a\}$
$f(S \cap T) = \varnothing \subset f(S) \cap f(T) = \{a\}$
So $a \in f(S) \cap f(T)$ but NO $x \in S \cap T$ with $f(x) = a$

## Solutions (Functions Supplement Problems)

1. Not one-to-one: $f(2) = f(2.38) = 2$
   Onto: Given → $z \in Z$, $z \in R$ and $f(z) = z$

2. One-To-One
   Let $x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2 \Rightarrow g(x_1) \neq g(x_2)$
   Not Onto: 1,3,5 ,... (all odd natural numbers) have no pre-images

3. One-To-One ***BIJECTION***
   Let $x_1 \neq x_2 \Rightarrow \frac{1}{x_1} \neq \frac{1}{x_2} \Rightarrow h(x_1) \neq h(x_2)$
   Onto: Given → $q \in Q^+, \frac{1}{q} \in Q^+$ and $h\left(\frac{1}{q}\right) = \frac{1}{\frac{1}{q}} = q$

4. Not one-to-one $g((8,2)) = g((4,1))$
   Onto: Given → $q \in Q^+, q = \frac{p_1}{p_2}$ Where $p_1, p_2 \in P$ (Definition of rational)
   So $(p_1, p_2) \in P \times P$ and $g((p_1, p_2)) = \frac{p_1}{p_2} = q$

5. Not one-to-one $f((2,2)) = f((4,0)) = 4$
   Onto: Given → $z \in Z$, $(z,0) \in Z \times Z$ and $f((z,0)) = z$

6. One-To-One ***BIJECTION***
   Let $s_1 \neq s_2 \Rightarrow 2s_1 + 1 \neq 2s_2 + 1 \Rightarrow f(s_1) \neq f(s_2)$
   Onto: Given → $n_0 \in N_0$, $n_0 = 2n + 1$ for some $n \in N$
   (Definition of odd integer)
   So $\frac{n_0 - 1}{2} \in N$ and $f\left(\frac{n_0 - 1}{2}\right) = n_0$

7. Not one-to-one $p(2) = p(3) = 0$
   Onto: $p(2) = 0$, $p(4) = 1$

8. Not one-to-one $k(-1) = k(1) = 1$
   Not Onto Negative Numbers have no pre-images

**Cardinality Supplement Problems**

1. Determine the cardinality of the following sets. (Use "unaccountably infinite" for sets not having cardinality $N_0$)

   a) $\{\sqrt{n} \mid n \in N\}$
   b) $\{z^2 + z \mid z \in Z\}$
   c) $\{\sqrt{n} \mid 1 \le n \le 100,\ n \in N\}$
   d) $\{n^2 \mid n \in N\}$
   e) $\{n \mid n \equiv 0 \mod 5,\ n \in Z\}$
   f) $\{n \mid n \equiv 0 \mod 10,\ n \in N\}$
   g) $\{z \mid -25 \le z \le 50,\ z \in Z\}$
   h) $Z \times Z$
   i) $\{r \mid 0 \le r \le 1,\ r \in R\}$

2. Prove the interval $(0,+\infty)$ is unaccountably infinite by showing $f: R \to (0,+\infty)$, $f(x) = e^x$ is bijection.

3. Prove the set of all sequences of 0's and 1's is unaccountably infinite, using Cantor's diagonal argument.

4. Prove if $S$ and $T$ are countable then $S \times T$ is countable.

5. Prove if $|S| = |T|$ then $|P(S)| = |P(T)|$, for all sets $S,T$.
   Hint: Find a bijection $g: P(S) \to P(T)$

**Solution Cardinality supplemental problems**

1. a) $N_o$, the set can be enumerated: $\{0,\sqrt{1},\sqrt{2},\sqrt{3},...\}$
   b) $N_o$, subsets of countable sets $(Z)$ are countable
   c) 100, set is finite $\{\sqrt{1},\sqrt{2},...,\sqrt{100}\}$
   d) $N_o$, the set can be enumerated $\{0,1,4,9,16,...\}$
      or subsets of countable sets (N) are countable
   e) $N_o$, the set can be enumerated $\{0,5,-5,10,-10,...\}$
      or subsets of countable sets (Z) are countable
   f) $N_o$, the set can be enumerated $\{0,10,20,30,...\}$ or subset of countable sets $(N)$ are countable
   g) 76, finite set $\{-25,-24...,0,1,...50\}$
   h) $N_o$, If Sand T are countable $(S=T=Z)$ then $S \times T$ is countable.
   i) Unaccountably infinite (Use Cantor's diagonal argument)

2. $f: R \to (o,+\infty)$, $f(x)=e^x$

   Function is one-to-one
   Let $x_1 \neq x_2$
   Then $e^{x_1} \neq e^{x_2}$
   So $f(x_1) \neq f(x_2)$

   Function is onto
   given $r \in (o,+\infty)$, $\ln r \in R$
   and $f(\ln r) = e^{\ln r} = r$

   Since f is a bijection, cardinality of Reals equals cardinality of $(o,+\infty)$. Since cardinality of Reals is unaccountably infinite, so is $(o,+\infty)$.

3. Assume the set of all sequences of 0's and 1's can be enumerated: $S_1, S_2, ... S_n, ...$
   List the values of each sequence:

   $S_1 = S_1(1), S_1(2), ..., S_1(n), ...$
   $S_2 = S_2(1), S_2(2), ..., S_2(n), ...$
   $\vdots$
   $S_n = S_n(1), S_n(2), ..., S_n(n), ...$

   $S_i(j) = jth$ value of sequence $S_i$
   $S_i(j) \in \{0,1\}$

Define $\overline{S} = \overline{S}(1), \overline{S}(2), \ldots, \overline{S}(n), \ldots$ by:

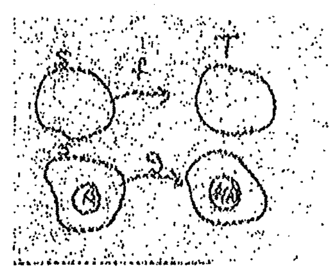
$$\overline{S}(i) = \begin{cases} 0 & S_i(i) = 1 \\ 1 & S_i(i) = 0 \end{cases} \quad \text{for } i = 1, 2, 3 \ldots.$$

$\overline{S}$ is a sequence of 0's and 1's and should be equal to one of the sequences on the list $S_1, S_2, \ldots, S_n, \ldots$. However, $\overline{S} \neq S_n$ for $n = 1, 2, \ldots$ because $\overline{S}(n) \neq S_n(n)$
(they differ at the $n^{th}$ value)
This contradicts the assumption that the set can be enumerated. Hence, the set of all sequences of 0's and 1's is unaccountably infinite.

**4.** Since S is countable, $Sx\{t\}$ is countable for $t \in T$ $(f : S \to Sx\{t\}$ by $f(s) = (s,t)$ is a *bijection*$)$
Then $SxT = \bigcup_{t \in T} Sx\{t\}$ is a countable union
(T is countable index) of countable sets $(Sx\{t\}$ is countable). By theorem, the countable union of countable sets is countable.

**5.** Since S and T have the same cardinality, there is a bijection f between S and T

Define a function $g : P(S) \to P(T)$ by
$g(A) = \{f(a) | \ a \in A\} = f(A)$




Prove g is a bijection:

One-to-one
Say $A_1 \neq A_2$, $A_1, A_2 \in P(S)$
There is an $a \in A_1$, $a \notin A_2$
But $f(a) \in f(A_1)$ and $f(a) \notin f(A_2)$
( since f is 1:1)
So $g(A_1) \neq g(A_2)$

Onto
Take $X \in P(T)$
$f^{-1}(X) = \{f^{-1}(x) | x \in X\}$ is an element of $P(S)$
$g(f^{-1}(X)) = \{f(f^{-1}(x)) | x \in X\} = X$
so $f^{-1}(X)$ is a preimage of $X$

## Review Exam I

1. Consider the set $A=\{\{1,2,3\},\{4,5\},\{6,7,8\}\}$
Determine whether each of the following is true or false
a) $1\in A$ b) $\{\{4,5\}\}\subset A$ c) $\{1,2,3\}\subset A$ d) $\varnothing\notin A$
e) $\{6,7,8\}\in A$

2. Answer true or false

a) For any set $A, A\subseteq P(A)$
b) $\varnothing\subseteq A$, for all sets A
c) $1\in\{\{1\},b,2\}$
d) $\{1,2\}\in P(A)$ where $A=\{1,2,3,4,5\}$
e) $\{1,b\}\subseteq\{\{1\},b,2\}$

3. Given $A=\{a,b,c,d,e\}$ $B=\{a,b,d,f,g\}$ $C=\{b,c,e,g,h\}$ $D=\{b,e,f,g,h\}$

a) $B-(C\cup D)$ b) $A\cap(B\cup D)$ c) $Ax(B-D)$ d) $P(A-(B\cap C))$

4. Prove $\overline{(A\cap B)\cup(\bar{A}\cap C)\cup(\bar{A}\cap B)}=(\bar{A}\cup\bar{B})\cap(A\cup(\bar{C}\cap\bar{B}))$

5. Let A,B be any sets

a) Prove $A=(A-B)\cup(A\cap B)$
b) Prove if $A\subset B$ then $A\cap\bar{B}=\varnothing$

6. For each function below, decide whether it is one-to-one, onto, both or neither. Prove your result.

a) $f:N\to N$ $f(n)=2n+1$
b) $g:Z\times Z\to Z$ $g(z_1,z_2)=z_1-z_2$
c) $h:Z\to Z$ $h(z)=z^2$
d) $k:R\to R$ $k(r)=r^3$

7. Decide whether each of the following relations are reflexive, irreflexive, symmetric, antisymmetric and / or transitive. Prove or disprove result.

$A = N = \{0,1,2,...\}$

a) $R = \{(x,y) | xy = n^2, n \in N\}$

b) $R = \{(x,y) | x > y\}$

8. Given a set A, define a relation on $P(A)$:

$R = \{(X,Y) | \overline{X} = Y\}$ *Answer as in (7)*

9. Define a relation R on Z, $R = \{(x,y) | x \equiv y \bmod 5\}$

Prove this is an equivalence relation and determine the equivalence classes.

10. For R a relation on A, prove

a) If R is reflexive, then so is $R^{-1}$

b) If R is symmetric, then so is $R^{-1}$

c) If R is transitive, then so is $R^{-1}$

11. Use Cantor's diagonal argument to show that the set of sequences of 1's, 2's and 3's is unaccountably infinite.

12. Determine the cardinality of the following sets (use "unaccountably infinite" for infinite sets not of cardinality $N_o$)

a) $\{\sqrt{n} | n \in N\}$

b) $\{z^2 + z | z \in Z\}$

c) $\{n | 50 \leq n, n \in N\}$

d) $\{z | z^2 - 2z = 0, z \in Z\}$

e) $\{r | 0 \leq r \leq 1, r \in R\}$

f) $Q \times Q$

g) $\{set \text{ of sequences of } 1's \text{ and } 2's\}$

h) $N \times R$

i) $R - P$

(h, i) State Theorems

**Solution Review Exam I (problems)**

1. a) F b) T c) F d) T e) T

2. a) F b) T c) F d) T e) F

3. a) $\{a, d\}$ b) $\{a,b,d,e\}$

c) $\{(a,a),(a,d),(b,a),(b,d),(c,a),(c,d),(d,a),(d,d),(e,a),(e,d)\}$

d) $\{\varnothing,\{a\},\{c\},\{d\},\{e\},\{a,c\},\{a,d\},\{a,e\},\{c,d\},\{c,e\},\{d,e\},\{a,c,d\},\{a,c,e\},$
$\{a,d,e\},\{c,d,e\},\{a,c,d,e\}\}$

4. $$\overline{(A\cap B)\cup(\overline{A}\cap C)\cup(\overline{A}\cap B)} \overset{Distributive}{=} \overline{(A\cap B)\cup(\overline{A}\cap(C\cup B))}$$

$$\overset{DeMorgan}{=} \left(\overline{A\cap B}\right)\cap\left(\overline{\overline{A}\cap(C\cup B)}\right) \overset{DeMorgan}{=} \left(\overline{A}\cup\overline{B}\right)\cap\left(\overline{\overline{A}}\cup\left(\overline{C\cup B}\right)\right)$$

$$\overset{DoubleComp, DeMorgan}{=} \left(\overline{A}\cup\overline{B}\right)\cap\left(A\cup\left(\overline{C}\cap\overline{B}\right)\right)$$

5. a) $(A-B)\cup(A\cap B) \overset{Thm\ A-B=A\cap\overline{B}}{=} \left(A\cap\overline{B}\right)\cup(A\cap B) \overset{Distributive}{=} A\cap\left(\overline{B}\cup B\right) \overset{Lemma}{=} A\cap U = A$

b) Assume $y\in A\cap\overline{B} \Rightarrow y\in A$ and $y\in\overline{B} \Rightarrow y\in A$ and $y\notin B$
this contradicts the assumption $A\subseteq B$

6. a) f is 1:1 If $x_1\neq x_2 \Rightarrow 2x_1+1\neq 2x_2+1 \Rightarrow f(x_1)\neq f(x_2)$
f is not onto: 0,2,... have no preimage

b) g is not 1:1 $g(2,1)=g(3,2)$
g is onto, Given $z\in Z$, $(z,0)\in Z\times Z$ and $g(z,0)=z$

c) h is not 1:1, $h(2)=h(-2)=4$
h is not onto, Negative integers, 2,3,5,6,7,... have no preimages.

d) k is 1:1 $r_1\neq 2_2 \Rightarrow r_1^3\neq r_2^3 \Rightarrow k(r_1)\neq k(r_2)$
k is onto, given $r\in R$, $r^{\frac{1}{3}}\in R$ and $f\left(r^{\frac{1}{3}}\right)^3 = r$

7. a) $R = \{(1,1),(1,4),(1,9),...(2,2],(2,8),...,(3,3),(4,1),(4,4),(4,9),...\}$
Reflexive: $(x,x) \in R$ for all $x \in N$ since $x \times x = x^2$
Not Irreflexive: $(1,1) \in R$
Symmetric If $(x,y) \in R$ then $xy = n^2$ or $yx = n^2$ or $(y,x) \in R$
for all $x, y \in N$, $n \in N$
Not Antisymmetric $(1,4) \in R$ $(4,1) \in R$
Transitive If $(x,y) \in R$ and $(y,z) \in R$ then $x \times y = n_1^2$, $y \times z = n_2^2$
Then $x \times z = \left(\frac{n_1 n_2}{y}\right)^2$

*Note.* $\frac{n_1 n_2}{y} \in N$ *(Proof by contradiction)* $a \nmid b = a$ *does not divide* $b$

*Say* $y \nmid n_1 n_2$ *then* $y^2 \nmid (n_1 n_2)^2$ *which contradicts* $xz = \left(\frac{n_1 n_2}{y}\right)^2 \in N$

b) $R = \{(2,1),(3,1),(3,2),...,(4,1),(4,2),...\}$
Not reflexive $(1,1) \notin R$
Irreflexive $(x,x) \notin R$ for all $x \in N$ since $x \not> x$
Not Symmetric $(2,1) \in R$, $2 > 1$ but $(1,2) \notin R$ $1 \not> 2$
Antisymmetric Impossible to have $(x,y) \in R$ and $(y,x) \in R$ since $x > y$
and $y > x$ impossible for $x, y \in N$
Transitive If $x, y \in R$ and $(y,x) \in R$ then $x > y$ and $y > z$ implies $x > z$
or $(x,z) \in R$

8. Not reflexive $\overline{X} \neq X$ for any set $X \in P(A)$
for example $A = \{a,b\}$ $P(A) = \{\varnothing,\{a\},\{b\},\{a,b\}\}$
$\overline{\{a\}} \neq \{a\}$ so $\{a\} \not R \{a\}$
Irreflexive $\overline{X} \neq X$ for all $X \in P(A)$, so $X \not R X$
Symmetric If $\overline{X} = Y$ then $\overline{\overline{X}} = \overline{Y}$ or $\overline{Y} = X$
Not Antisymmetric $\{a\} R \{b\}$ and $\{b\} R \{a\}$ but $\{a\} \neq \{b\}$
Not transitive $\{a\} R \{b\}$, $\{b\} R \{a\}$ but $\{a\} \not R \{a\}$

9. $x R x$ since $x - x = 0 = 5 \times 0$
If $x R y$, $x - y = 5k$, $y - x = 5(-k)$ or $y R x$ $(k \in z)$
If $x R y$, $y R z$, $x - y = 5k_1$, $y - z = 5k_2$, then $x - z = 5(k_1 + k_2)$
$k_1, k_2 \in Z$.

$[0] = \{...,-5,0,5,10,...\}$ $[1] = \{...,-4,1,6,11,...\}$ $[2] = \{...,-3,2,7,...\}$
$[3] = \{...,-2,3,8,13...\}$ $[4] = \{...,-1,4,9,14,...\}$

10. a) For all $x \in A$, $(x,x) \in R \Rightarrow (x,x) \in R^{-1}$ $(Def\ \ of\ \ R^{-1})$

b) $(x,y) \in R^{-1} \xrightarrow{defR^{-1}} (y,x) \in R \xrightarrow{Rsym} (x,y) \in R \xrightarrow{defR^{-1}} (y,x) \in R^{-1}$

c) $(x,y) \in R^{-1}$ and $(y,z) \in R^{-1} \xrightarrow{defR^{-1}} (y,x) \in R$ and $(z,y) \in R$

$\xrightarrow{Rtrans} (z,x) \in R \xrightarrow{defR^{-1}} (x,z) \in R^{-1}$

11. Assume the set of all sequences of 1's, 2's and 3's can be enumerated: $S_1, S_2, \ldots, S_n, \ldots$ where

$$S_1 = S_1(1), S_1(2), \ldots, S_n(n), \ldots$$
$$S_2 = S_2(1), S_2(2), \ldots, S_n(n), \ldots$$
$$\vdots$$
$$S_n = S_n(1), S_n(2), \ldots, S_n(n), \ldots$$

$S_i(j) = j^{th} element\ \ of\ S_i$
$S_i(j) \in \{1,2,3\}$

Define $\overline{S} = \overline{S}(1), \overline{S}(2), \ldots \overline{S}(n), \ldots$ by

$$\overline{S}(i) = \begin{cases} 1 \ldots if \ldots S_i(i) = 2,3 \\ 2 \ldots if \ldots S_i(i) = 1 \end{cases} \qquad \text{for } i = 1,2,\ldots$$

This is a sequence of 1's,2's and 3's and should be on the list of sequences $S_1, S_2, \ldots, S_n, \ldots$
however, $\overline{S} \neq S_n$ for $n = 1,2,\ldots$ because $\overline{S}(n) \neq S_n(n)$ (the $n^{Th}$ value of $S_n$
there fore the set of all sequences of 1's,2's and 3's cannot be enumerated and is uncountably infinite.

12. a) No $\{\sqrt{1}, \sqrt{2}, \ldots\}$ set can be enumerated

b) No $\{0,2,6,\ldots\}$ set can be enumerated

c) No $\{50,51,52,\ldots\}$ set can be enumerated

d) 2

e) Uncountably infinite (Cantor's diagonal argument)

f) No If S and T are countable then $S \times T$ is countable

g) Uncountably infinite ( Cantor's diagonal argument)

h) Uncountably infinite $IR \cong \{0\} \times IR \subseteq N \times IR$

If A is uncountably infinite and $A \subseteq B$ then B is uncountably infinite

i) Uncountably infinite If A is uncountably infinite and B is countable then $A - B$ is uncoutably infinite

## Logic Supplementary Problems

1. Determine whether the following arguments are valid or invalid. Give counterexample or prove validity with a truth table.

a)
$$\begin{array}{l} r \to s \\ p \to g \\ \underline{r \vee p} \\ \therefore s \vee g \end{array}$$

b)
$$\begin{array}{l} p \to (r \to s) \\ \neg r \to \neg p \\ \underline{p} \\ \therefore s \end{array}$$

c)
$$\begin{array}{l} \neg r \\ p \to q \\ \underline{q \to r} \\ \therefore \neg p \end{array}$$

d)
$$\begin{array}{l} \neg p \\ p \to q \\ \underline{q \to r} \\ \therefore \neg r \end{array}$$

2. If Tallahassee is not in Florida, then golf balls are not sold in Chicago. Golf balls are not sold in Chicago. Hence Tallahassee is in Florida.

3. If the cup is Styrofoam, then it is lighter than water. If the cup is lighter than water, then Joe can carry it. Therefore if the cup is Styrofoam then Joe can carry it.

4. If wagers are raised, buying increases. If there is a depression, wages cannot be raised. Thus if there is a depression, buying cannot increase.

5. The given triangles are similar. If the given triangles are mutually equiangular, then they are similar. Therefore, the triangles are mutually equiangular.

## Solution

1. a) Valid show truth table $[(r \to s) \wedge (p \to q) \wedge (r \vee p)] \to (s \vee q)$ is a tautology

   b) Valid show truth table of $[(p \to (r \to s)) \wedge (7r \to 7p) \wedge p] \to s$ is a tautology.

   c) Valid show truth table of $[(\neg r) \wedge (p \to q) \wedge (q \to r)] \to 7r$ is a tautology.

   d) Invalid $p = F$ $q = TorF$ $r = T$

2. $\neg p \to \neg q$

   $\neg q$

   $\therefore p$

   Invalid $p = F$ $q = F$

   p= Tallahassee..FL
   q= golf balls sold in Chicago

3. $p \to q$

   $q \to r$

   $\therefore p \to r$

   Valid show $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology

   p= cup Styrofoam
   q= lighter water
   r= Joe carries it

4. $p \to q$

   $r \to \neg p$

   $\therefore r \to \neg q$

   Invalid $p = F$, $q = T$, $r = T$

   p= wages raised
   q=buying increased
   r= depression

5. $p$

   $q \to p$

   $\therefore q$

   Invalid $p = T$ $g = F$

   p= triangles similar
   q=triangles mutually equiangular

**Induction and Recursion**
**Supplementary Problems**

1. Define the Fibonacci sequence: $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$
prove $f_n < \left(\frac{7}{4}\right)^n$ for $n = 1,2,\ldots$

2. Recursively define $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$

   a) Calculate $a_2, a_3, a_4, a_5$

   b) Guess a formula for $a_n$ (explicitly)

   c) Prove your guess

3. Recursively define $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$

   a) Calculate $a_2, a_3, a_4, a_5$

   b) Guess a formula for $a_n$ (explicitly)

   c) Prove your guess

4. Recursively define $a_0 = a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$for $n \geq 3$

   a) Calculate $a_3, a_4, a_5 a_6$

   b) Prove that $a_n \leq 2^{n-1}$ for $n = 1,2,\ldots$

5. Recursively define $b_0 = b_1 = b_2 = 1$; $b_n = b_{n-1} + b_{n-3}$ for $n \geq 3$

   a) Calculate $b_3, b_4, b_5, b_6$

   b) Prove that $b_n \geq \left(\sqrt{2}\right)^{n-2}$ for $n \geq 2$

6. Prove $n^5 - n$ is divisible by 10 for $n \in N$

7. Prove $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for $n \in N$

## Solutions

1. Basis cases $f_1 = 1 < \left(\frac{7}{4}\right)^1$, $f_2 = 2 < \left(\frac{7}{4}\right)^2$

Assume $f_i < \left(\frac{7}{4}\right)^i$ for $i = 1,2,\ldots,k$

Then $f_{k+1} = f_k + f_{k-1} < \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}+1\right) < \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{k+1}$

2. a) $a_2 = 3,\ a_3 = 4,\ a_4 = 5,\ a_5 = 6$

b) Guess $a_n = n+1,\ n = 0,1,2,\ldots$

c) Basis cases $a_{0rec} = 1 \quad a_{0\exp} = 0+1 = 1$

$a_{1rec} = 2 \quad a_{1\exp} = 1+1 = 2$

Assume $a_{irec} = a_{i\exp} = i+1$ for $i = 0,1,2,\ldots k$

Then $a_{k+1rec} = 2a_{krec} - a_{k-1rec} = 2a_{k\exp} - a_{k-1\exp}$

$= 2(k+1) - (k-1+1) = k+2 = a_{k+1\exp}$

3. a) $a_2 = 4, a_3 = 8, a_4 = 16, a_5 = 32$

b) Guess $a_n = 2^n \quad n = 0,1,2,\ldots$

c) Basis cases $a_{0rec} = 1 = 2^0 = a_{0\exp}$

$a_{1rec} = 2 = 2^1 = a_{1\exp}$

Assume $a_{irec} = a_{i\exp} = 2^i$ for $i = 0,1,2,\ldots k$

Then $a_{k+1rec} = a_{krec} + 2a_{k-1rec} = a_{k\exp} + 2a_{k-1\exp}$

$= 2^k + 2 \times 2^{k-1} = 2^k + 2^k = 2 \times 2^k = 2^{k+1} = a_{k+1\exp}$

4. a) $a_3 = 3, a_4 = 5, a_5 = 9, a_6 = 17$

b) Basis cases $a_1 = 1 \le 2^0 \qquad a_3 = 3 \le 2^2 = 4$

$a_2 = 1 \le 2^1$

Assume $a_i \le 2^{i-1}$ for $i = 1,2,\ldots k$

Then $a_{k+1} = a_k + a_{k-1} + a_{k-2} \le 2^{k-1} + 2^{k-2} + 2^{k-3}$

$= 2^{k-3}\left[2^2 + 2 + 1\right] \le 2^{k-3}\left(2^3\right) = 2^k$

## Solutions continued

5. a) $b_3 = 2, b_4 = 3, b_5 = 4, b_6 = 6$

b) $b_2 \geq (\sqrt{2})^0$ or $1 \geq 1$

$b_3 \geq (\sqrt{2})^1$, or $2 \geq \sqrt{2}$ } Basis cases

$b_4 \geq (\sqrt{2})^2$ or $3 \geq 2$

Assume $b_i \geq (\sqrt{2})^{i-2}$ for $i = 2,3,\ldots k$

Then $b_{k+1} = b_k + b_{k-2} \geq (\sqrt{2})^{k-2} + (\sqrt{2})^{k-4}$

$$= (\sqrt{2})^{k-4}\left(\sqrt{2}^2 + 1\right) = (\sqrt{2})^{k-4}(3) \geq (\sqrt{2})^{k-4}(\sqrt{2})^3$$

$\left(\text{since}\,(3) \geq (\sqrt{2})^3\right)$

$$= (\sqrt{2})^{k-1}$$

6. Basis case $n = 0$ $\quad 0^5 - 0 = 0 = 10.0$

Assume $n = k$, $k^5 - k$ is divisible by 10

Then $(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$

$$= k^5 - k + 10(k^3 + k^2) + 5k(k^3 + 1)$$

$k^5 - k$ is divisible by 10 by assumption

$10\,(k^3 + k^2)$ has a factor of 10

$5k(k^3 + 1)$ is divisible by 5

If k is even then $5k(k^3 + 1)$ is divisible by 10 (Since k divisible by 2)

If k is odd, $k^3$ is odd and $k^3 + 1$ is even,

So $5k(k^3 + 1)$ is divisible by 10 (Since $k^3 + 1$ divisible by 2)

There fore $(k+1)^5 - (k+1)$ is divisible by 10

7. Basis case $n = 0$ $\quad 5^1 + 2(3)^0 + 1 = 8$ is divisible by 8

Assume $n = k$: $\quad 5^{k+1} + 2 \times 3^k + 1$ is divisible by 8

Then $5^{k+2} + 2 \times 3^{k+1} + 1 = 5\left(5^{k+1} + 2 \times 3^k + 1\right) + 3^k(-10 + 6) - 4$

$$= 5\underbrace{\left(5^{k+1} + 2 \times 3^k + 1\right)}_{\text{divisible by 8 by assumption}} - 4\underbrace{\left(3^k + 1\right)}_{\text{divisible by 8}}$$

## Combinatorics Supplementary Problems

1. How many eight-bit strings begin 1100?

2. How many eight-bit strings begin and end with 1?

3. How many eight-bit strings have either the second or the fourth bit 1 (or both)?

4. How many eight-bit strings have exactly one 1?

5. How many eight-bit strings have exactly two 1's?

6. How many eight-bit strings have at least one 1?

7. How many eight-bit strings read the same from either end ? (An example of such an eight-bit string is 01111110 )

*For 8 – 11 Consider 5 Computer science books, 3 English books, 2 Arts books.*

8. In how many ways can these books be arranged on a shelf?

9. In how many ways can these books be arranged on a shelf if all five computer science books are on the left and both art books are on the right?

10. In how many ways can these books be arranged on a shelf if all five computer science books are on the left?

11. In how many ways can these books be arranged on a shelf if all books of the same discipline are grouped together?

*For 12 – 14 refer to a club consisting of six men and seven women.*

12. In how many ways can we select a committee of three men and four women?

13. In how many ways can we select a committee of four persons which has at least one woman?

14. In how many ways can we select a committee of four persons that has at most one man?

15. Give factorial arguments for the 4 combinatorial identities.
    a) $C(n, r) = C(n, n\text{-}r)$ b) $P(n, r) = nP(n\text{-}1, r\text{-}1)$
    c) $C(n, r) = C(n\text{-}1, r)+C(n\text{-}1, r\text{-}1)$
    d) $C(n,r)\times C(r,k) = C(n,k)\times C(n\text{-}k,r\text{-}k)$

16. Find the coefficient of

a) $w^6z^6$ in the expansion of $(2w^3 - 3z^2)^5$
b) $s^4$ in the expansion of $(2s+5)^7$

17. Show that $0 = \sum_{k=0}^{n} (-1)^k c(n,k)$

18. Show that $3^n = \sum_{k=0}^{n} 2^k c(n,k)$

19. Use the result from #17 to prove that the number of odd subsets equals the number of even subsets.

20. Find the number of (unordered) 13 – cards bridge hands selected from an ordinary 52 – cards deck.

21. How many bridge hands are all of the same suit?

22. How many bridge hands contain exactly two suits?

23. How many bridge hands contain all four aces?

24. How many bridge hands contain five spades, four hearts, three clubs, and one diamond?

*For 25 – 28 Determine how many strings can be formed by ordering the letters ABCDE subject to the condition given.*

25. A appears before D. (Examples: BCAED, BCADE)

26. A and D are side by side. (Example, ADBCE, BCDAE)

27. The patterns AB and CD do not appear.

28. The patterns AB and BE do not appear.

29. In how many ways can six persons be seated around a circular table?

30. In how many ways can six keys be put on a ring? (Turning the ring over does not count as a different arrangement)

31. In how many ways can five Martians and five Jovians wait in line?

32. In how many ways can five Martians and five Jovians wait in line if no two Martians stand together?

**33.** In how many ways can five Martians and five Jovians be seated at a circular table?

**34.** In how many ways can five Martians and five Jovians be seated at a circular table if no two Martians sit together?

**35.** In how many ways can five Martians and eight Jovians wait in line if no two Martians stand together?

**36.** In how many ways can five Martians and eight Jovians be seated at a circular table so that no two Martians sit together?

## Solutions -- Combinatorics

**1)** $2^4$ **2)** $2^6$ **3)** $3\times 2^6$ **4)** 8 **5)** $C(8,2)=28$ **6)** $2^8-1$

**7)** $2^4$ **8)** $10!$ **9)** $5!3!2!$ **10)** $5!5!$ **11)** $5!3!2!3!$ **12)** $C(6,3)C(7,4)$

**13)** $C(13,4)-C(6,4)$ **14)** $C(7,4)+C(7,3)C(6,1)$

**15)** a) $C(n,r)=C(n,n-r)$

$$\frac{n!}{r!(n-r)!}=\frac{n!}{(n-r)![n-(n-r)]!}$$

b) $P(n,r)=nP(n-1,r-1)$

$$\frac{n!}{(n-r)}=\frac{n(n-1)}{[(n-1)-(r-1)]!}$$

$$=\frac{n!}{(n-r)}$$

c) $C(n,r)=C(n-1,r)\quad +C(n-1,r-1)$

$$\frac{n!}{r!(n-r)!}=\frac{(n-1)!}{r!(n-1-r)!}+\frac{(n-1)!}{(r-1)!(n-r)!}=\frac{(n-1)!(n-r)+(n-1)!r}{r!(n-r)!}=\frac{(n-1)!(n-r+r)}{r!(n-r)!}$$

$$=\frac{n(n-1)!}{r!(n-r)!}=\frac{n!}{r!(n-r)!}$$

d) $C(n,r)C(r,k) = C(n,k)C(n-k,r-k)$

$$\frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{(r-k)!(n-r)!}$$

16) a) $C(5,3)2^2(-3)^3$ b) $C(7,3)2^4 5^3$

17) put $x = 1$ and $y = -1$ into the binomial theorem

18) Put $x = 1$ and $y = 2$ into the binomial theorem

19) $0 = \sum_{k=0}^{n}(-1)^k C(n,k) \rightarrow 0 = C(n,o) - C(n,1) + C(n,2) - \ldots \pm C(n,n)$

Note: last term is positive if even, negative if odd.

$C(n,1) + \ldots + C(n,n-1) = C(n,o) + C(n,2) \ldots + C(n,n)$

assuming n even.

20. $C(52,13)$ 21) $4C(13,13) = 4$ 22) $C(4,2)(C(26,13) - 2)$

23) $C(48,9)$ 24) $C(13,5) \times C(13,4) \times C(13,3) \times C(13,1)$

25) $5!/2$ 26) $2 \times 4!$ 27) $5! - 3!$ 28) $5! - 3!$

29) $5!$ 30) $5!/2$ 31) $10!$ 32) $P(6,5)5!$

33) $9!$ 34) $5!4!$ 35) $8!P(9,5)$ 36) $7!P(8,5)$

# Review-Exam II

1. Prove by induction that

a) $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+.....+\frac{n}{2^n}=2-\frac{(n+2)}{2^n}$ for $n \in P$

b) $1\cdot 1!+2\cdot 2!+.....+n\cdot n!=(n+1)!-1$ for $n \in P$

c) $(1+x)^n \geq 1+nx$ for $n \in P$ $x$ a real number, $x \geq -1$

2. How many 4 digit numbers can be formed using the digits 2,3,5,6,8,9
a) if repetitions are allowed
b) if no repetitions are allowed
c) if those in (b) are even numbers
d) if those in (b) are greater than 4000
e) if those in (b) are divisible by 5

3. Consider the set S = {a, e, i, b, c, d, f, g, h, m, n, p}
How many 5 letter words containing 2 different vowels and 3 different consonants?
a) can be formed?
b) contain the letter "b"?
c) begin with "a"?

4. A committee of k people is to be chosen from a set of 9 women and 5 men. How many ways to form the committee if the committee has
a) 6 people, 3 women and 3 men
b) any number of people, but equal numbers of men and women
c) 6 people and at least 3 are women
d) 6 people including Mr. A (one of the men)
e) 6 people but Mr. & Mrs. A cannot both be on the committee

5. Use the binomial theorem to prove

$$[C(n,0)+.....+C(n,n)]^2=(\sum_{k=0}^{n}C(n,k))^2=\sum_{k=0}^{2n}C(2n,k)$$

6. Use a factorial argument to prove $C(n+2, r) = C(n+1,r)+C(n+1, r-1)$

7. Find the coefficient of $x^9y^{10}$ in $(3x^3-4y^2)^8$

8. Use mathematical induction to prove that the values of the explicit function $g(n) = 5 \cdot 2^n + 1$ are the same as the function defined recursively by $g(0) = 6$, $g(1) = 11$, $g(n) = 3g(n-1) - 2g(n-2)$ for $n \geq 2$.

9. Construct a truth table for $(p \vee q) \to (\neg p \vee \neg r)$

Translate into symbolic language using p, q, r and determine the validity of the following arguments.

10. If I study, then I will not fail mathematics. If I do not play basketball, then I will study. But I failed mathematics. Therefore, I played basketball.

11. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 classes, how many different rooms will be needed?

12.
a) Write the negation of "No one has lost more than $1000 playing the lottery".

b) Write the negation of "Some student has solved all the exercises".

c) Write the negation of: $\forall x \exists y (P(x) \wedge Q(y))$ (Use De Morgans)

# Solutions for Review Exam II

1. a) $n=1$ $\quad \frac{1}{2}=2-\frac{3}{2}=\frac{1}{2}$

Assume $n=k$: $\quad \frac{1}{2}+\ldots\ldots+\frac{k}{2^k}=2-\frac{(k+2)}{2^k}$

$$\frac{1}{2}+\ldots\ldots+\frac{k}{2^k}+\frac{k+1}{2^k}=2-\frac{(k+2)}{2^k}+\frac{k+1}{2^{k+1}}$$
$$=2+\frac{-2(k+2)+k+1}{2^{k+1}}$$
$$=2-\frac{(k+3)}{2^{k+1}}$$

b) $n=1$ $\quad 1\cdot 1!=2!-1=1$

Assume $n=k$ $\quad 1\cdot 1!+\ldots\ldots+k\cdot k!=(k+1)!-1$

$$1\cdot 1!+\ldots\ldots+k\cdot k!+(k+1)(k+1)!=(k+1)!-1+(k+1)(k+1)!$$
$$=(k+1)!(1+(k+1))-1$$
$$=(k+1)!(k+2)-1$$
$$=(k+2)!-1$$

c) $n=1$ $\quad (1+x)^1\geq 1+x$

Assume $n=k$: $\quad (1+x)^k\geq 1+kx$

$$(1+x)^{k+1}=(1+x)^k(1+x)\geq(1+kx)(1+x)=1+kx+kx^2+x$$
$$=(1+(k+1)x+kx^2\underbrace{\geq}_{\text{since } kx^2\geq 0}1+(k+1)x$$

2. a) $6^4$ $\quad$ b) $P(6,4)=\frac{6!}{2!}=360$ $\quad$ c) $3\cdot P(5,3)=3\cdot\frac{5!}{2!}$

d) $4\cdot P(5,3)=\frac{4\cdot 5!}{2!}$ $\quad$ e) $1\cdot P(5,3)$

3. a) $C(3,2)C(9,3)\cdot 5!$ $\quad$ b) $C(3,2)C(8,2)\cdot 5!$ $\quad$ c) $C(2,1)C(9,3)\cdot 4!$

4. a) $C(9,3)C(5,3)$

b) $C(9,1)C(5,1)+C(9,2)C(5,2)+C(9,3)C(5,3)+C(9,4)C(5,4)+C(9,5)C(5,5)$

c) $C(9,3)C(5,3)+C(9,4)C(5,2)+C(9,5)C(5,1)+C(9,6)$

d) $C(13,5)$

e) $\underbrace{C(12,5)}_{\text{Mr. A on, Mrs. A off}}+\underbrace{C(12,5)}_{\text{Mrs. A on, Mr. A off}}+\underbrace{C(12,6)}_{\text{both off}}=C(14,6)-\underbrace{C(12,4)}_{\text{both Mr. \& Mrs. A on}}$

5. $(1+1)^n=\sum_{k=0}^{n}C(n,k)$ $\qquad$ $(1+1)^{2n}=\sum_{k=0}^{2n}C(2n,k)$

by plugging into binomial theorem $x=y=1$ (and $2n \to n$ in second case)

But,

$$(1+1)^{2n}=[(1+1)^n]^2=\left(\sum_{k=0}^{n}C(n,k)\right)^2=(C(n,0)+\ldots\ldots+C(n,n))^2$$

6.

$$\frac{(n+2)!}{r!(n+2-r)!}=\frac{(n+1)!}{r!(n+1-r)!}+\frac{(n+1)!}{(r-1)!(n+2-r)!}=\underbrace{\frac{(n+1)!(n+2-r)+(n+1)!r}{r!(n+2-r)!}}_{\text{least common denominator}}$$

$$=\frac{(n+1)!(n+2)}{r!(n+2-r)!}=\frac{(n+2)!}{r!(n+2-r)!}$$

7. $C(8,3)\cdot 3^3(-4)^5$

8. Basis cases $n=0,1$ $\qquad$ $g(0)_{rec}=6$ $\qquad$ $g(0)_{exp}=5\cdot 2^0+1=6$

$g(1)_{rec}=11$ $\qquad$ $g(1)_{exp}=5\cdot 2^1+1=11$

Assume $g(i)_{rec}=g(i)_{exp}=5\cdot 2^i+1$ for $i=0,1,2,\ldots\ldots k$

$$g(k+1)_{rec}=3g(k)_{rec}-2g(k-1)_{rec}\underbrace{=}_{\text{by assumption}}3g(k)_{exp}-2g(k-1)_{exp}$$

$$=3(5\cdot 2^k+1)-2(5\cdot 2^{k-1}+1)=15\cdot 2^k+3-5\cdot 2^k-2$$

$$=10\cdot 2^k+1=5\cdot 2^{k+1}+1=g(k+1)_{exp}$$

9.

| $p$ | $q$ | $r$ | $(p \vee q) \to (\neg p \vee \neg r)$ |
|---|---|---|---|
| T | T | T | F |
| T | T | F | T |
| T | F | T | F |
| T | F | F | T |
| F | T | T | T |
| F | T | F | T |
| F | F | T | T |
| F | F | F | T |

10.

| $p$ | $q$ | $r$ | $[(p \to \neg q) \wedge (\neg r \to p) \wedge q] \to r$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | T |
| F | T | T | T |
| F | T | F | T |
| F | F | T | T |
| F | F | F | T |

Therefore, argument is valid.

11. $\left\lceil \frac{677}{38} \right\rceil = 18$

12. a) Someone has lost more than \$1000 playing the lottery.

b) All students have not solved all the exercises.

c) $\exists x \forall y (\neg P(x) \vee \neg Q(y))$

## Graph Theory Supplement Problems

**1.** How many vertices will the following graphs have if they contain (may be pseudographs)

a) 16 edges and all vertices of degree 2

b) 21 edges, 3 vertices of degree 4 and the other vertices of degree 3

c) 24 edges and all vertices of the same degree

**2.** determine if the graph exists. If so, draw a representation.

a) 6 vertices each of degree 3

b) 5 vertices each of degree 3

c) 4 vertices each of degree 1

d) 6 vertices, 4 edges

e) 4 edges; 4 vertices having degrees 1,2,3,4

f) 4 vertices having degrees 1,2,3,4

**3.** Given the following digraph

a) Compute the adjacency matrix A

b) Compute the adjacency matrix $A^2$


**4.** Compute a digraph corresponding to the adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Compute $A^2$

## Solutions

**1.** a) $2n = 2(16) \Rightarrow n = 16$

b) $3(4) + k(3) = 2(21) \Rightarrow k = 10 \qquad k + 3 = 13$ (total)

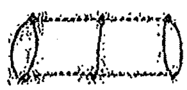
c) $n = \#\, vertices$ k= degree of vertex $\quad n \times k = 24(2) = 48$

Different possibilities

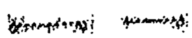
$n = 1 \quad k = 48; \quad n = 2 \quad k = 24; \quad n = 3 \quad k = 16; \quad n = 4 \quad k = 12$

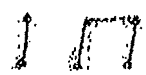
$n = 6 \quad k = 8; \quad n = 8 \quad k = 6; \quad n = 12 \quad k = 4; \quad n = 16 \quad k = 3$

$n = 24 \quad k = 2; \quad n = 48 \quad k = 1$

**2.** a) 

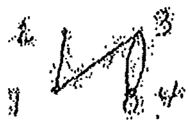
b) No graph: odd number of odd degree vertice

c) 

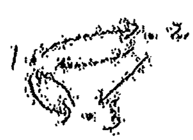
d) 

e) No graph $4(2) = 8$

$1 + 2 + 3 + 4 = 10$

f) 

**3.** $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

**4.**  $A^2 = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

**5.** There are 3 paths of length 3 from $v_1$ to $v_3$

**6.** $\begin{pmatrix} 3 & 11 & -4 \\ 13 & 3 & 1 \\ 8 & 2 & 1 \end{pmatrix}$

5. For a graph, the adjacency matrix $A^3$ is as follows $\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 4 & 0 & 2 & 0 \end{pmatrix}$ What does "3" in the (1,3) position mean?

6. Multiply $\begin{pmatrix} 1 & 5 & 0 \\ -1 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & -1 \\ 4 & 0 & 1 \end{pmatrix}$

## Solutions – even numbered problems

P649 4. Multigraph    6. Multigraph    8. Directed multigraph

P665 2. $|V|=5$, $|E|=13$; $\deg(a)=6$, $\deg(b)=6$, $\deg(c)=6$, $\deg(d)=5$, $\deg(e)=3$, no isolated or pendant points

8. $|V|=4$, $|E|=8$, $\deg^-(a)=2$, $\deg^-(b)=3$, $\deg^-(c)=2$, $\deg^-(d)=1$, $\deg^+(a)=2$, $\deg^+(b)=4$, $\deg^+(c)=1$, $\deg^+(d)=1$.

P675 6.

$$A=\begin{pmatrix}0&1&0&1&0\\1&0&0&1&1\\0&0&0&1&1\\1&1&1&0&0\\0&1&1&0&0\end{pmatrix}$$

8.

$$\begin{pmatrix}0&1&0&1&0\\1&0&1&1&1\\0&1&1&0&0\\1&0&0&0&1\\0&0&1&0&1\end{pmatrix}$$

14.

$$\begin{pmatrix}0&3&0&1\\3&0&1&0\\0&1&0&3\\1&0&3&0\end{pmatrix}$$

20.

$$\begin{pmatrix}1&1&1&1\\0&1&0&1\\1&0&1&0\\1&1&1&1\end{pmatrix}$$

p675

35. Isomorphic $u_1\to v_1$, $u_2\to v_3, u_3\to v_5, u_4\to v_2, u_5\to v_4$
$\{u_1,u_2\}\to\{v_1,v_3\},\{u_2,u_3\}\to\{v_3,v_5\},\{u_3,u_4\}\to\{v_5,v_2\}$
$\{u_4,u_5\}\to\{v_2,v_4\},\{u_5,u_1\}\to\{v_4,v_1\}$
or show adjacency matrices same, reordering vertices in second graph.

36. Not isomorphic. The second graph has a vertex of degree 4, whereas the first graph does not.

37. Isomorphic $u_1\to v_1, u_2\to v_3, u_3\to v_5, u_4\to v_7, u_5\to v_2, u_6\to v_4, u_7\to v_6$.
$\{u_1,u_1\}\to\{v_1,v_3\}$ etc. show image of every edge is an edge, or show adjacency matrices same.

41. Not isomorphic. In the first graph the vertices of degree 3 are adjacent to a common vertex. This is not the case in the second graph.

42. Not isomorphic. Vertices of degree 4 are adjacent in first graph. Not so in second graph.

p675

61. Isomorphic $u_1 \to v_3, u_2 \to v_4, u_3 \to v_2, u_4 \to v_1$
$(u_2,u_1) \to (v_4,v_3), (u_2,u_2) \to (v_4,v_4), (u_1,u_3) \to (v_3,v_2)$
$(u_3,u_1) \to (v_2,v_3), (u_3,u_2) \to (v_2,v_4), (u_3,u_4) \to (v_2,v_1)$
$(u_2,u_4) \to (v_4,v_1)$ or show adjacency matrices same

62. Not isomorphic
$\deg^-(u_1) = 0$, $\deg^+(u_1) = 2$. Under isomorphism,
$u_1 \to v_3$ (only vertex with same in out degree structure).
But $u_1$ is adjacent to two vertices with indegree 1. And $v_3$ is adjacent to two vertices, one with indegree 1, and another of indegree 2.

63. Isomorphic $u_1 \to v_3, u_2 \to v_1, u_3 \to v_4, u_4 \to v_2$
$(u_1,u_3) \to (v_3,v_4), (u_2,u_1) \to (v_1,v_3), (u_2,u_3) \to (v_1,v_4)$
$(u_3,u_4) \to (v_4,v_2), (u_4,u_2) \to (v_2,v_1), (u_1,u_4) \to (v_3,v_2)$
or show adjacency matrices are the same.

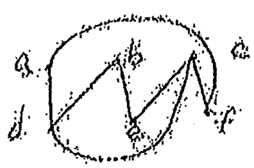
p703 2. *a d g h i f c b e f h e d b a* (circuit)

4. *f a b c e a d b f e d c*

6. *b c d e f d g i d a h i a b i c*

14. Has Euler path (has exactly 2 odd vertices)

p725 6.



p732 4. 2 6. 3 8. 3

p789 10. *d b f e g a c*

24. a) 32 b) 40 c) 32

# Review Exam III

1. A graph with 21 edges has 7 vertices of degree 1, 3 of degree 2, 7 of degree 3 and the rest of degree 4. How many vertices in graph?

2. Can the following degree structure occur in a simple graph? If so, draw a representation. If not, explain.

   a) 1 vertex degree 0, 1vertex degree1, 3 vertices degree 3, 1 vertex degree 4

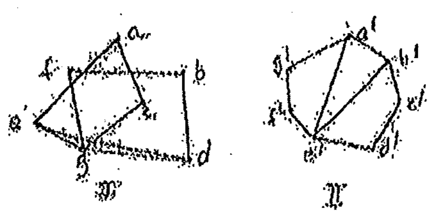
   b) 2 vertices degree 2, 2 vertices degree 3, 1 vertex degree 4

3. A tree has 3 vertices degree2, 2 vertices degree 3, and 1vertex degree 4. If the remaining vertices have degree 1, how many vertices in the tree?

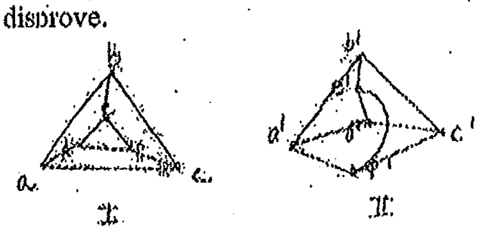
4. Prove that the sum of degrees of the vertices in the tree with $n$ vertices is $2n-2$

5. Are the following graphs isomorphic? Prove or disprove.

a)

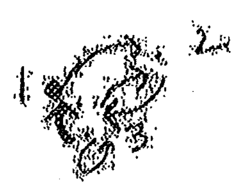


b)



6. For the following digraph, determine the adjacency matrix

$A$ and $A^2$:

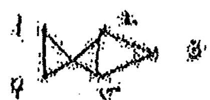


7. Give a representation of the following digraph given its adjacency matrix:

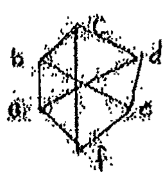
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Is the digraph strongly connected, weakly connected or not connected?

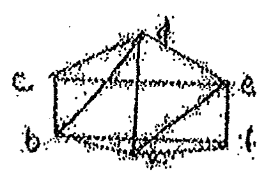
8. Find the number of paths of length 2 between vertex 3 and vertex 4 using matrices:



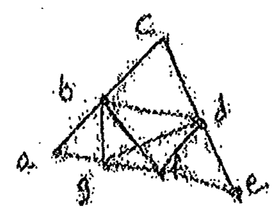
9. Answer parts $a$, $b$, $c$ for the following 3 graphs



(*i*)



*ii*



*iii*

a) Decide if each graph has an Euler path. If so, indicate the path. If not explain.

b) Draw a planar representation of each graph, if possible

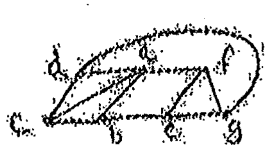
c) Determine the chromatic number of each graph

10. a) Represent the following logical expression as a tree and rewrite in polish prefix notation:

$$(\neg(p \vee q)) \vee (\neg p \vee (q \wedge r))$$

b) Represent the following postfix notation as a tree and write in standard notation.
$pqr \vee 7 \wedge qr \vee \wedge$

11. Find three subgraphs that are trees using all the vertices.

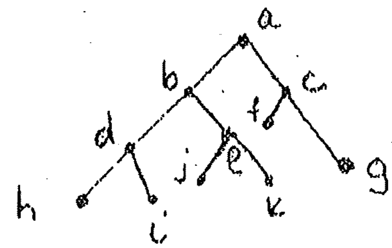


**12.** Show the sequential orders in which the vertices of the tree are visited in

a) Preorder traversal

b) Inorder traversal

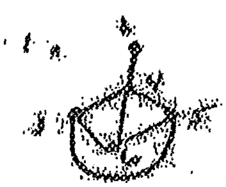
c) Postorder Traversal



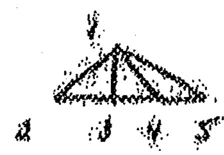

## Solution

**1.** $42 = 7(1) + 3(2) + 7(3) + n(4)$ where $n$ = number of vertices degree 4
2 vertices degree 4, 19 vertices in graph

**2.** a)



b)



**3.** $3(2) + 2(3) + 1(4) + n(1) = 2\ (|V| - 1) = 2((6 + n) - 1) = 10 + 2n$
where $n$ = number of vertices of degree 1, $n = 6$, 12 vertices in graph

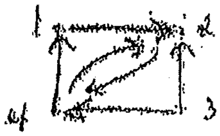
**4.** $\sum_{k=1}^{n} \deg(v_k) = 2|E| = 2(|V| - 1) = 2(n-1) = 2n - 1$

**5.** a) Isomorphic

$a \to c'$ $\{a,c\} \to \{c',d'\}$
$b \to g'$ $\{c,g\} \to \{d',e'\}$
$c \to d'$ etc.
$d \to a'$ check all 9 edges
$e \to b'$
$f \to f'$
$g \to e'$

b) Non isomorphic
$2K_3$'s in I, no $K_3$ in II

6. a). $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$    b) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

7. 

weakly connected    (Path from 3 to 2 but none from 2 to 3)

8.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 2 & \textcircled{1} & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

There is one path between 3 and 4 of length 2

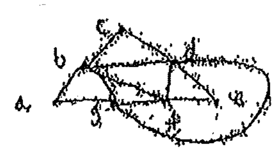
9. *i*) None
Too many odd degree Vertices

*ii*) *c d e f a b c e a d b f*    *iii*) *b c d e f g a b d f b g d*

*i*) Non planar ($K_{3,3}$)

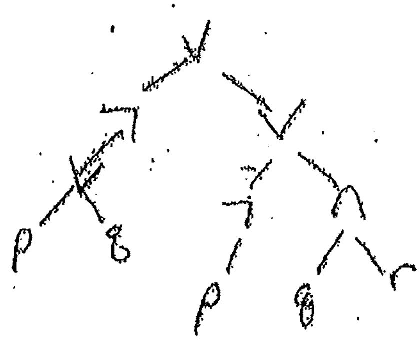
*ii*) Planar

*iii*) Planar

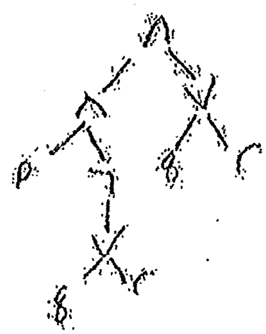


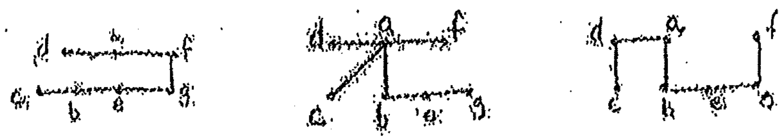
*i*) 2    *ii*) 3    *iii*) 4.

10. a)




$\vee \neg \vee pq \vee \neg p \wedge qr$

b)  $(p \wedge \neg(q \vee r)) \wedge (q \vee r)$

**11.**  (there are many others)

**12.**

*a b d h i e j k c f g*

*h d i b j e k a f c g*

*h i d j k e b f g c a*

## Boolean Algebras

**1.** Show that any Boolean algebra, for all $a,b \in B$

a) If $a \bullet b = a$ then $a + b = b$

b) If $a + b = b$ then $\bar{a} + b = 1$

c) If $\bar{a} + b = 1$ then $a \bullet \bar{b} = 0$

d) If $a \bullet \bar{b} = 0$ then $a \bullet b = a$

**2.** Find the sum of products form for each of the following Boolean functions and use a Karnaugh map to find an equivalent minimal sum of products form.

a) $f(x,y,z) = xy + (x+y)\bar{z} + y$

b) $f(x,y,z) = (x + y + \bar{z})(\bar{x} + y + \bar{z})$

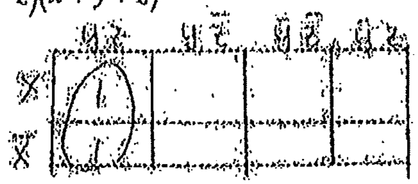
c) $f(x,y,z,w) = y\bar{z}w + x\bar{z}\bar{w} + xy\bar{z}w + x\bar{y}\,\bar{z}w$

## Solutions to even numbered problems

p 827

2) $\overline{\bar{x} \bullet \bar{y}}$

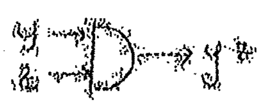
4) $(\overline{\bar{x} \bullet y \bullet z})(\bar{x} + y + \bar{z})$



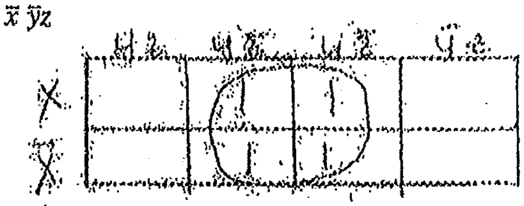
p 841

6) a) $f(x,y,z,) = xyz + \bar{x}yz$

$f(x,y,z,) = yz$



b) $f(x,y,z,) = xy\bar{z} + x\bar{y}\,\bar{z} + \bar{x}y\bar{z} + \bar{x}\,\bar{y}z$

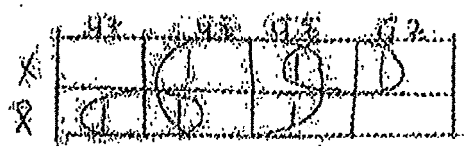


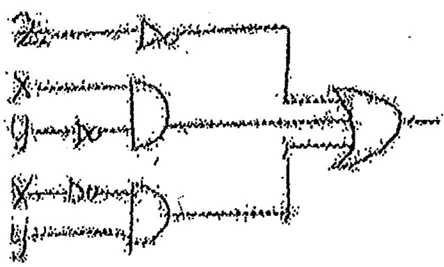
$f(x,y,z,) = \bar{z}$

$z \rightarrow\triangleright\circ\rightarrow \bar{z}$

c) Change function to $f(x,y,z,) = \bar{x}yz + (x+\bar{z})(\bar{y}+\bar{z})$

$f(x,y,z,) = \bar{z} + x\bar{y} + \bar{x}y$





$f(x,y,z)$

**12.** a) $\bar{x}z$

b) $y$

c) $x\bar{z} + \bar{y}z + \bar{x}z$
or
$x\bar{z} + x\bar{y} + \bar{x}z$

d) $yz + \bar{x}\,\bar{z} + x\bar{y}$
or
$xz + \bar{x}y + \bar{y}\,\bar{z}$

**Solution Supplement Problems**

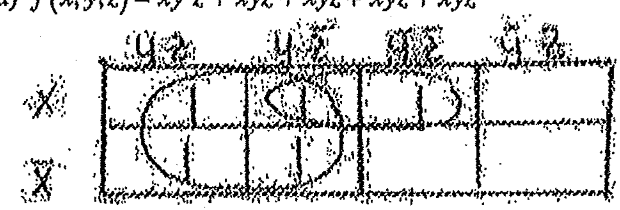
**1.** a) $a+b=ab+b$ (Assumption)$=a\bullet b+1\bullet b$ (Identity)$=(a+1)\bullet b$ (distributive)
$=1\bullet b(\textit{theorem } a+1=1)=b(\textit{identity})$

b) $\bar{a}+b=\bar{a}+(a+b)$ $(\textit{Assumption})$ $=(\bar{a}+a)+b$ $(\textit{Associative})$
$1+b(\textit{complement})=1$ $(\textit{theorem } a+1=1)$

c) $a\bullet\bar{b}=(a\bullet\bar{b})\bullet 1=(\textit{Identity})=(a\bullet\bar{b})(\bar{a}+b)$ $(\textit{Assumption})$
$=(a\bullet\bar{a})\bullet\bar{b}+a\bullet(\bar{b}\bullet b)$ (Distributive, Commutative, Associative)
$=0\bullet\bar{b}+a\bullet 0$ $(\textit{complement})$
$=0+0$ $(\textit{theorem } a\bullet 0=0)$
$=0$ $(\textit{Identity})$

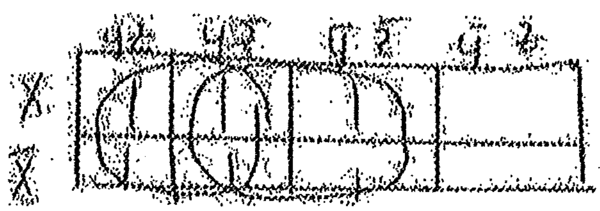
d) $a\bullet b=a\bullet b+0$ $(\textit{Identity})=a\bullet b+a\bullet\bar{b}$ $(\textit{Assumption})$
$=a(b+\bar{b})(\textit{Distributive})$ $=a(1)$ $(\textit{complement})=$ a $(\textit{Identity})$

**2.** a) $f(x,y,z)=x\bar{y}\,\bar{z}+\bar{x}yz+xyz+\bar{x}y\bar{z}+xy\bar{z}$



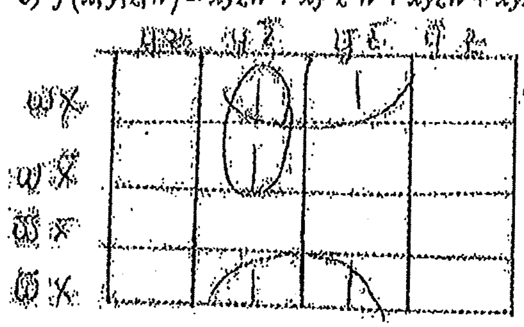
$f(x,y,z)=y+x\bar{z}$

b) $f(x,y,z)=\bar{x}\,\bar{y}\,\bar{z}+x\bar{y}\,\bar{z}+\bar{x}yz+xyz+\bar{x}y\bar{z}+xy\bar{z}$



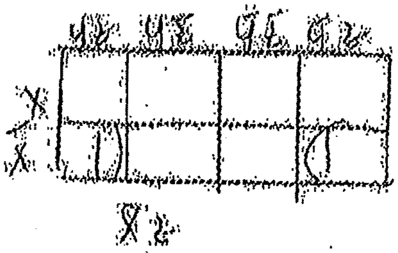
$f(x,y,z)=y+\bar{z}$

c) $f(x,y,z,w)=xy\bar{z}\bar{w}+x\bar{y}\,\bar{z}\,\bar{w}+\bar{x}y\bar{z}w+xy\bar{z}w+x\bar{y}\,\bar{z}w$

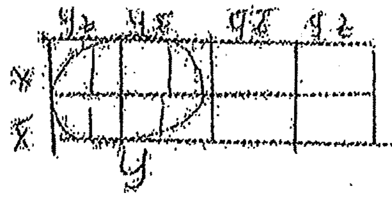


$f(x,y,z,w)=x\bar{z}+y\bar{z}w$

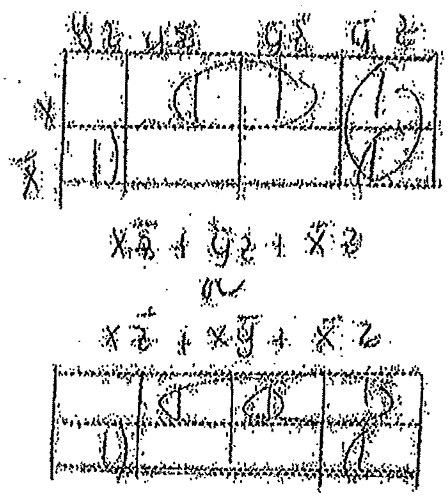
a)



b)



c)



d)