Combinatorics.

Techniques for counting large finite sets without listing elements.

Ex How many ways for a salesman to visit 50 cities?

$\frac{50}{1} \times \frac{49}{2} \times \frac{48}{3} \times \cdots \times \frac{1}{50} = 50! \approx 3 \times 10^{64}$

Computer makes 10^9 additions / second

10^{47} years to perform additions!

<u>Thm</u> the Sum Rule.

If a first task can be done in n_1 ways and a second task can be done in n_2 ways and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task

In general, if $X = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \varphi$

(disjoint union of $A_1, A_2 \ldots, A_n$) then

$|X| = \sum_{i=1}^{n} |A_i|$

Ex. Cards 52 in deck. 4 suits - hearts, clubs, spades and diamonds. 13 kinds: A, 2-10, J, Q, K.

1. In how many ways can we draw a heart or a spade? $13+13=26$
2. a heart or an ace? $13+3=12+4=16$
3. an ace or a king? $4+4=8$
4. a card numbered 2 through 10? $9 \times 4 = 36$

Ex Die - cube with faces of 1-6 dots
Roll 2 dice.

5. How many ways to get a 7 or 11?

7		11	
4	3	6	5
3	4	5	6
5	2		
2	5		
1	6		
6	1		

$6+2=8$

Thm The Product Rule.

Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task (after the first task has been done) then there are $n_1 \cdot n_2$ ways to do the procedure.

S_1 then S_2 then ... then S_n

$|S_1 \times S_2 \times \cdots \times S_n| = \prod_{i=1}^{n} |S_i|$

1. Chairs in an auditorium are labelled with a letter followed by a positive integer less than or equal to 100. How many chairs can be labelled differently? $26 \times 100 = 2600$

2. How many bit strings of length n?
$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$

3. How many different license plates have 3 letters followed by 4 digits?
a) repetition allowed $26^3 \times 10^4$
b) only letters repeated $26^3 \times 10 \times 9 \times 8 \times 7$
c) no repetitions $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$

4. Telephone Numbering Plan

X 0-9
Y 0, 1
N 2-9

Old plan: NYX- NNX-XXXX
New plan: NXX- NXX-XXXX

Old: $8\times 2\times 10\times 8\times 8\times 10\times 10^4 = 1{,}024{,}000{,}000$

New: $8\times 10\times 10\times 8\times 10\times 10\times 10^4 = 6{,}400{,}000{,}000$

5. Computer system has password 6-8 characters (letter or digit)

a) How many passwords? $36^6+36^7+36^8$

b) How many passwords if each password must have at least one digit?

$36^6+36^7+36^8-(26^6+26^7+26^8) = 2{,}684{,}483{,}063{,}360$

Thm Pigeonhole Principle

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing 2 or more of the objects

Proof (Contradiction) Suppose none of the k boxes contains more than 1 object. Then the total number of objects would be at most k. Contradiction.

① Among 102 students taking an exam (0-100) at least 2 students must have the same grade

② Among any group of 367 people, at least 2 people must have the same birthday (counting Feb 29) thm The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing $\lceil \frac{N}{k} \rceil$ objects

where $\lceil x \rceil$ is the smallest integer $\geq x$.

Note: $\lceil \frac{N}{k} \rceil < \frac{N}{k}+1$

$\lceil 2.5 \rceil = 3 \quad \lceil -2 \rceil = -2 \quad \lceil -2.5 \rceil = -2$

$\lceil 2.5 \rceil = 3 < 3.5$

$\lceil -2 \rceil = -2 < -1$

$\lceil -2.5 \rceil = -2 < -1.5$

Proof Contradiction

Suppose all of the boxes contain no more than $\lceil \frac{N}{k} \rceil - 1$ objects. Then the total number of objects is at most $k\left(\lceil \frac{N}{k} \rceil - 1\right) < k\left(\frac{N}{k}+1-1\right) = N$

Contradiction

① Among 100 people, at least $\lceil \frac{100}{12} \rceil = 9$ people were born in the same month.

② What is the minimum number of students needed to guarantee at least 6 students will receive the same grade? (A, B, C, D, F) $k=5$

$\lceil \frac{N}{5} \rceil = 6 \qquad N = 5(6-1)+1 = 26$

In general $\lceil \frac{N}{k} \rceil = L \qquad$ Minimum $N = k(L-1)+1$

③ How many area codes are needed in an area with 25 million phones to assure each number is distinct?

NXX-XXXX $\quad N\ 2\text{-}9 \quad X\ 0-10$

8×10^6 = 7 digit phone numbers

$\lceil \frac{25 \times 10^6}{8 \times 10^6} \rceil = 4$ (one number will be used 4 times so need 4 area codes)

Permutations

Def A permutation of r objects is an <u>ordered</u> arrangement of these objects

EX $S = \{a, b, c\}$

$r = 3$ permutations
abc
acb
bac
bca
cab
cba

$r = 2$ permutations
ab
ba
ac
ca
bc
cb

Thm the number of r-permutations of a set with n distinct elements is.

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

In particular, there are $P(n,n) = n!$ permutations of n objects

EX1. Suppose there are 8 runners in a race. Winner receives a gold medal, 2nd place silver, 3rd place bronze. How many different ways to award medals if all outcomes are possible?

$P(8,3) = 8 \times 7 \times 6$

EX2. In how many ways can 7 women and 3 men

a) be arranged in a row? $10!$

b) so that women are together and men together?

$2 \times 7! \, 3!$

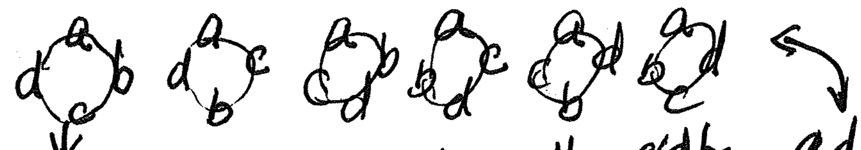
c) so that men are together? $3! \, 8!$

d) so that the men are separated from each other? $7! \, P(8,3)$

Thm there are $(n-1)!$ circular arrangements of n objects

Proof Fix one object. Cut the circle counterclockwise to that object. Form a linear arrangement. Delete the fixed object. this will give you a bijection between circular arrangements of n objects and linear arrangements of $n-1$ objects

Ex



~~a~~bcd ~~a~~cbd ~~a~~bcd ~~a~~cdb ~~a~~dbc ~~a~~dcb

bijection between circular arrangement of 4 objects and linear arrangement of 3 objects

Combinations

Def An r- combination of elements from a set is an *unordered* selection of r elements (r subset).

$S = \{a,b,c,d\}$ $r=3$ $\{a,b,c\}$ $\{a,b,d\}$ $\{b,c,d\}$ $\{a,c,d\}$

thm the number of r combinations from a set with n elements is $C(n,r) = \dfrac{n!}{r!(n-r)!}$

Proof For each r combination from a set with n elements, we can form $r!$ r-permutations.

So $C(n,r)\cdot r! = P(n,r)$ or

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$$

Ex. 1 How many 5 card hands? $C(52,5)$

a) of hearts? $C(13,5)$

b) from a single suit? $C(4,1)C(13,5)$

c) have 2 aces and 3 kings? $C(4,2)\,C(4,3)$

d) have 2 of one kind and 3 of another kind?

$P(13,2)\,C(4,2)\,C(4,3)$

$= 2\,C(13,2)\,C(4,2)\,C(4,3)$

$= 13 \times 12 \times C(4,2)\,C(4,3)$

Ex 2. There are 21 consonants, 5 vowels $\{a,e,i,o,u\}$ in alphabet.

Consider 8 letter strings, no repeats

a) How many strings? $P(26,8)$

b) How many 8 letter strings with 3 vowels and 5 consonants?

$C(5,3)\,C(21,5) \times 8!$

i) containing "a" $C(4,2)\,C(21,5) \times 8!$

ii) containing "a, b" $C(4,2)\,C(20,4) \times 8!$

iii) containing "a", but not "b" $C(4,2)C(20,5) \times 8!$

iv) starting with "a", ends with "b" $C(4,2)C(20,4) \times 6!$

Combinatorial Identities

See homework problem 15 in Combinatorics Supplementary problems for solutions. Prove

① $C(n,r) = C(n, n-r)$

② $C(n,r)\,C(r,k) = C(n,k)\,C(n-k, r-k)$

③ $P(n,r) = n\,P(n-1, r-1)$

④ Pascal's Identity $C(n,r) = C(n-1,r) + C(n-1,r-1)$

Pascal's Triangle

Note: $C(n,0) = C(n,n) = 1$

$C(0,0)$

$C(1,0)$ $C(1,1)$

$C(2,0)$ $C(2,1)$ $C(2,2)$

$C(3,0)$ $C(3,1)$ $C(3,2)$ $C(3,3)$

$C(4,0)$ $C(4,1)$ $C(4,2)$ $C(4,3)$ $C(4,4)$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

For interior values, use Pascal's identity

Binomial Theorem

<u>Thm</u> the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k$$

$$= C(n,0)x^n + C(n,1)x^{n-1}y + \dots + C(n,n-1)xy^{n-1} + C(n,n)y^n$$

① Expand $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Expand $(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$

② Find the coefficient of

a) s^7t^5 in $(s+t)^{12}$ $n=12$

$C(12,k)s^{12-k}t^k$ $k=5$: $C(12,5)s^7t^5$

b) s^7t^5 in $(2s-t)^{12}$ $n=12$

$C(12,k)(2s)^{12-k}(-t)^k$ $k=5$ $C(12,5)2^7(-1)^5$

c) x^8y^{18} in $(2x^2+3y^3)^{10}$ $n=10$

$C(10,k)(2x^2)^{10-k}(3y^3)^k$ $k=6$

$C(10,6)2^4 3^6$

③ Prove $2^n = \sum_{k=0}^{n} C(n,k)$

Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} C(n,k) x^{n-k} y^k$

let $x=y=1$ $\quad (1+1)^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} 1^k$

$$2^n = \sum_{k=0}^{n} C(n,k)$$

<u>Proof</u> of binomial theorem

$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + \cdots + C(n,n-1)xy^{n-1} + C(n,n)y^n$

for $n = 1, 2, \ldots$

$n=1 \quad (x+y)^1 = C(1,0)x^1 + C(1,1)y^1 = x+y$

Assume

$(x+y)^k = C(k,0)x^k + C(k,1)x^{k-1}y + \cdots + C(k,k)y^k$

Prove

$(x+y)^{k+1} = C(k+1,0)x^{k+1} + \cdots + C(k+1,k+1)y^{k+1}$

Multiply assumption by $(x+y)$

Pascal's Identity

$(x+y)^k(x+y) = C(k,0)x^{k+1} + \underline{C(k,0)x^k y + C(k,1)x^k y} + C(k,1)x^{k-1}y^2 + \cdots + C(k,k)xy^k + C(k,k)y^{k+1}$

$C(k,0) = C(k,k) = 1 = C(k+1,0) = C(k+1,k+1)$

$C(k+1,0)x^{k+1} + C(k+1,1)x^k y + \cdots\cdots C(k+1,k+1)y^{k+1}$

(Middle terms collapse into appropriate terms)