Cardinality

Let's say A and B are finite sets

If $|A| > |B|$, is there a function $f: A \to B$ which is 1:1? $A = \{1,2,3\}$ $B = \{a,b\}$

$$\begin{aligned} 1 &\xrightarrow{f} a \\ 2 &\to b \\ 3 &\to ? \end{aligned}$$

NO

If $|A| < |B|$, is there a function $f: A \to B$ which is onto? $A = \{1,2\}$ $B = \{a,b,c\}$

$$\begin{aligned} 1 &\xrightarrow{f} a \\ 2 &\to b \ c \ ? \end{aligned}$$

NO

So for finite sets A and B, if there is a bijection $f: A \to B$ then $|A| = |B|$

Def Define a relation, $\cong$, "isomorphism" on the set of all sets:

$A \cong B$ if there is a bijection $f: A \to B$

(A is isomorphic to B)

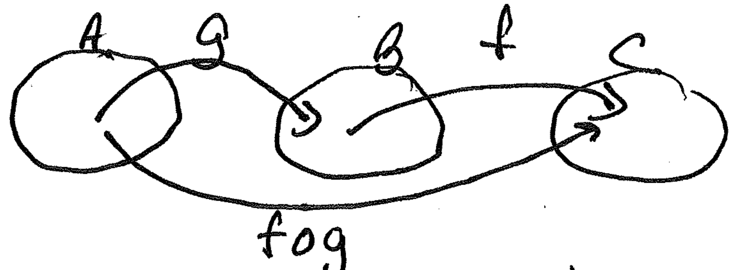
Thm. Isomorphism relation, "$\cong$" is an equivalence relation on the set of all sets.

Reflexive: $i_A: A \to A$ by $i_A(x) = x$ is a bijection so $A \cong A$ for all sets A.

Symmetric: If $A \cong B$ then $B \cong A$ for all sets A, B

Since $A \cong B$ there is a bijection $f: A \to B$ then $f^{-1}: B \to A$ is a bijection. So $B \cong A$

Transitive: If $A \cong B$ and $B \cong C$ then $A \cong C$ for all sets A, B, C



Since $A \cong B$ there is a bijection $g: A \to B$
Since $B \cong C$ there is a bijection $f: B \to C$
We need to prove $A \cong C$, we need to prove $f \circ g: A \to C$ is a bijection. To do so:

<u>thm</u> If $g: A \to B$ and $f: B \to C$ are one-to-one functions then $f \circ g: A \to C$ is a one-to-one function

Proof: $a_1 \neq a_2$, $a_1, a_2 \in A$

$g(a_1) \neq g(a_2)$ (g is one-to-one)

$f(g(a_1)) \neq f(g(a_2))$ (f is one-to-one)

$f \circ g(a_1) \neq f \circ g(a_2)$ (Def $f \circ g$)

<u>thm</u> If $g: A \to B$ and $f: B \to C$ are onto functions then $f \circ g: A \to C$ is an onto function.

Proof. Take $c \in C$. Since f is an onto function, there is a $b \in B$ with $f(b) = c$. Since g is an onto function there is an $a \in A$ with $g(a) = b$. So

$$f \circ g(a) = f(g(a)) = f(b) = c$$

Isomorphism is an equivalence relation on the set of all sets, and therefore the set of all sets can be partitioned into equivalence classes.

Def $|P| = \aleph_0$ (aleph null) $P = \{1, 2, 3, \dots\}$

Thm $N = \{0, 1, 2, \dots\}$ $|N| = \aleph_0$

Proof $f : P \to N$ $f(p) = p - 1$ is a bijection

Thm $Z = \{\dots -2, -1, 0, 1, 2, \dots\}$ $|Z| = \aleph_0$

Proof $f : Z \to P$ $f(z) = 2z+1$ if $z \geq 0$ is a bijection.
$= -2z$ if $x < 0$

$$\begin{array}{lcccccccccc}
Z: & \dots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \dots \\
P: & & & & 1 & 2 & 3 & 4 & 5 &
\end{array}$$

Thm $|Q| = \aleph_0$ Q = rationals.

$$\begin{array}{ccccccccccc}
\dots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \dots \\
\dots & -2 & -\frac{3}{2} & \cancel{-1} & -\frac{1}{2} & & \frac{1}{2} & \cancel{1} & \frac{3}{2} & 2 & \dots \\
 & -\frac{4}{3} & -1 & -\frac{2}{3} & -\frac{1}{3} & & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \dots \\
 & \vdots & & & & \vdots & & & \vdots & &
\end{array}$$

Proof:
Every rational number is in the above infinite matrix.
Follow the arrows and skip repetitions

$P = \{1, 2, 3, 4, \dots\}$
$Q = \{0, 1, \frac{1}{2}, -\frac{1}{2}, \dots\}$ $f : P \to Q$ is a bijection

Thm $|P \times P| = \aleph_0$

$P = \{1, 2, 3, 4, \dots\}$
$P \times P = \{(1,1), (1,2), (2,2), \dots\}$

Proof
$$\begin{array}{llll}
(1,1), & (1,2), & (1,3), & \dots \\
(2,1), & (2,2), & (2,3), & \dots \\
(3,1), & (3,2), & (3,3), & \dots \\
\vdots & & &
\end{array}$$

Follow arrows $f : P \to P \times P$ is a bijection

Def. A <u>countably infinite</u> set is a set with cardinality $\aleph_0$

A <u>countable</u> set is one which is either <u>finite</u> or <u>countably infinite</u>

Def An enumeration of A is a listing of A: $A = (a_1, a_2, \ldots, a_n, \ldots)$ which lists A without repetition and has one a_n for each $n \in P$ (must be an infinite set - one for each $n \in P$)

Thm If a set can be enumerated, the set has cardinality $\aleph_0$.

Proof. $A = (a_1, a_2, \ldots, a_n, \ldots)$ (definition of enumeration)

$f: P \to A$ by $f(n) = a_n$ is a bijection

Thm Subsets of countable sets are countable.

Proof Prove subsets of P are countable.

Let $A \subseteq P$

If A is finite, it is countable.

Suppose A is infinite:

$A = (a_1, a_2, a_3, \ldots)$ where a_1 = least element

a_2 = least element in $A - \{a_1\}$

etc

Thm The countable union of countable sets is countable.

$\bigcup_{i \in I} A_i$ Let I = countable index set.

Either $I = \{1, 2, \ldots, n\}$ or $I = P$

a) let $I = \{1, 2, \ldots, n\}$ $\quad a_{ij} = j^{th}$ element of set A_i

$$\bigcup_{i=1}^{n} A_i = \{a_{11}, a_{21}, \ldots, a_{n1}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots \}$$

is an enumeration of the union (if infinite)

(each A_i is countable - assume at least one A_i is infinite)

b) let $I = P$

$\bigcup_{i=1}^{+\infty} A_i$ we will show it can be enumerated:

$$\begin{array}{ll} A_1: & a_{11} \rightarrow a_{12} \quad a_{13} \cdots \\ & \qquad\quad \downarrow \\ A_2: & a_{21} \leftarrow a_{22} \quad a_{23} \cdots \\ & \\ A_n: & a_{n1} \quad a_{n2} \quad a_{n3} \cdots \\ & \vdots \end{array}$$

$\bigcup_{i=1}^{\infty} A_i$ can be enumerated

(follow arrows and leave out repeats)

<u>thm.</u> If S and T are countable then $S \times T$ is countable.

Proof: Supplement homework exercise.

<u>Ex.</u> $|N \times P| = \aleph_0$ $\quad |\{1,2,3\} \times Q| = \aleph_0$ $\quad |Z \times \{a, \ldots, z\}| = \aleph_0$

<u>Def</u> An infinite set which is not countable is called uncountably infinite.

<u>Thm</u>. the real numbers are uncountably infinite (Cantor's Diagonal Argument).

<u>Proof</u> Assume the real numbers are countebly infinite and can be enumerated:

$\mathbb{R} = (r_1, r_2, \dots, r_n, \dots)$

$r_1 = a_1 . b_{11} b_{12} \cdots b_{1n} \cdots$

$r_2 = a_2 . b_{21} b_{22} \quad b_{2n} \cdots$

$\vdots$

$r_n = a_n . b_{n1} b_{n2} \quad b_{nn} \cdots$

$\vdots$

a_i = integer portion of r_i

b_{ij} = j^{th} decimal of r_i

$b_{ij} \in \{0, 1, 2, \dots 9\}$

Define a real number, $\overline{b} = .\overline{b_{11}}\,\overline{b_{22}} \cdots \overline{b_{nn}} \cdots$

where $\overline{b_{ii}} = \begin{cases} 0 & \text{if } b_{ii} = 1,2,3,4,5,6,7,8,9 \\ 1 & \text{if } b_{ii} = 0 \end{cases}$ $\quad i = 1, 2, 3, \dots$

$\overline{b}$ is a real number and should be on the list $\mathbb{R} = (r_1, r_2, \dots, r_n, \dots$

but $\overline{b} \neq r_n$ for $n = 1, 2, 3, \dots$ because $\overline{b_{nn}} \neq b_{nn}$ (they are different at the n^{th} decimal place)

Contradiction: the real numbers cannot be enumerated and are uncountably infinite.

Note: Why does the above argument not work for the rational numbers?

$\overline{b} = .\overline{b_{11}}\,\overline{b_{22}} \cdots \overline{b_{nn}} \cdots$ is a real number, but we can't be sure it is a rational number.

EX Prove $[0,1) = \{r \mid 0 \le r < 1, r \in \mathbb{R}\}$ is uncountably infinite:

Use same argument as with the real numbers

Change $a_i = 0$

EX Prove $[1,2) = \{r \mid 1 \le r < 2, r \in \mathbb{R}\}$ is uncountably infinite:

Use same argument as with the real numbers

Change $a_i = 1$

$$\bar{b} = 1.\bar{b}_{11}\,\bar{b}_{22} \cdots \bar{b}_{nn} \cdots$$

EX. Prove the set of all sequences of a's, b's & c's is uncountably infinite:

Assume the set is countably infinite and can be enumerated: $S = (s_1, s_2, \dots, s_n, \dots)$

$s_1 = s_1(1), s_1(2), \dots, s_1(n), \dots$

$s_2 = s_2(1), s_2(2), \dots, s_2(n), \dots$

$\vdots$

$s_n = s_n(1), s_n(2), \dots, s_n(n), \dots$

$\vdots$

$s_i(j) =$ j^{th} value of sequence s_i

$s_i(j) \in \{a, b, c\}$

Define a sequence $\bar{s} = \bar{s}(1), \bar{s}(2), \dots, \bar{s}(n), \dots$

where
$$\bar{s}(i) = \begin{cases} a & \text{if } s_i(i) = b \\ b & \text{if } s_i(i) = c \\ c & \text{if } s_i(i) = a \end{cases} \qquad i = 1, 2, 3, \dots$$

$\bar{s} \ne s_n$ for $n = 1, 2, \dots$ since $\bar{s}(n) \ne s_n(n)$ (different at n^{th} value)

Contradiction: the set of all sequences of a's, b's & c's cannot be enumerated and is uncountably infinite.

Ex Prove the set of sequences of 0's & 1's is uncountably infinite

Change previous argument $S_i(j) \in \{0, 1\}$

$$\bar{S}(i) = \begin{cases} 0 & \text{if } S_i(i) = 1 \\ 1 & \text{if } S_i(i) = 0 \end{cases} \quad i = 1, 2, 3, \ldots$$

Thm. If A is uncountably infinite and $A \subseteq B$ then B is uncountably infinite.

Proof Contradiction. Assume B is countable. Since $A \subseteq B$ then A is countable (subsets of countable sets are countable). Contradiction since A is uncountably infinite.

Ex Prove $\mathbb{R} \times \mathbb{Z}$ is uncountably infinite.

$$\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq \mathbb{R} \times \mathbb{Z}$$
$$A \subseteq B$$

If A is uncountably infinite (isomorphic to $\mathbb{R}$) and $A \subseteq B$ then B is uncountably infinite.

Thm. If A is uncountably infinite and B is countable then $A - B$ is uncountably infinite.

Proof Contradiction. Assume $A - B$ is countable. then $(A-B) \cup B$ is countable (a countable union of countable sets is countable). But $A \subseteq (A-B) \cup B$ ($x \in A \Rightarrow (x \in A$ and $x \notin B)$ or $x \in B$). So A is countable since subsets of countable sets are countable. Contradiction.