Boolean Algebra

Def A boolean algebra $(\mathcal{B}, +, \cdot, \bar{\ }, 0, 1)$ is a set, $\mathcal{B}$, with two binary operators, $+$, $\cdot$, one unary operator, $\bar{\ }$, and two elements $0, 1 \in \mathcal{B}$ with the following properties:

1. There exists at least two elements $a, b \in \mathcal{B}$ with $a \neq b$.
2. $\forall a, b \in \mathcal{B}$, $a+b \in \mathcal{B}$, $a \cdot b \in \mathcal{B}$ (Closure)
3. $\forall a, b \in \mathcal{B}$, $a+b = b+a$, $a \cdot b = b \cdot a$ (Commutative)
4. $\forall a \in \mathcal{B}$, $a+0 = a$, $a \cdot 1 = a$ (Identity)
5. $\forall a, b, c \in \mathcal{B}$ $a \cdot (b+c) = a \cdot b + a \cdot c$ (Distributive)
   $a + (b \cdot c) = (a+b) \cdot (a+c)$
6. $\forall a \in \mathcal{B}$, $\exists \bar{a} \in \mathcal{B}$ with $a + \bar{a} = 1$ (Complement)
   $a \cdot \bar{a} = 0$
7. $\forall a, b, c \in \mathcal{B}$ $a + (b+c) = (a+b)+c$ (Associative)
   $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Boolean algebras we have studied:

$\mathcal{B}$	$\{T, F\}$	$P(X)$
$+$	$\vee$	$\cup$
$\cdot$	$\wedge$	$\cap$
$\bar{\ }$	$\neg$	$\overline{(\ )}$
0	F	φ
1	T	$\mathcal{U}$

Principle of Duality

Every property or theorem has a dual property or theorem (interchange $+$ and $\cdot$, interchange 0 and 1).

Theorems for all boolean algebras

Thm $\forall a \in \beta \quad a+a=a \qquad$ Dual $(a \cdot a = a)$

$$
\begin{aligned}
a+a &= (a+a)\cdot \underline{1} && \text{Identity} \\
&= (a+a)\cdot(a+\bar{a}) && \text{Complement} \\
&= a + a\cdot\bar{a} && \text{Distributive} \\
&= a + 0 && \text{Complement} \\
&= a && \text{Identity}
\end{aligned}
$$

thm $\forall a \in \beta \quad a+1=a \qquad$ Dual $(a\cdot 0 = 0)$

$$
\begin{aligned}
a+1 &= (a+1)\cdot \underline{1} && \text{Identity} \\
&= (a+1)\cdot(a+\bar{a}) && \text{Complement} \\
&= a + 1\cdot\bar{a} && \text{Distributive} \\
&= a + \bar{a} && \text{Identity} \\
&= 1 && \text{Complement}
\end{aligned}
$$

thm $\forall a \in \beta \quad a+a\cdot b=a \qquad$ Dual $a\cdot(a+b)=a$

$$
\begin{aligned}
a+a\cdot b &= a\cdot\underline{1} + a\cdot b && \text{Identity} \\
&= a\cdot(1+b) && \text{Distributive} \\
&= a\cdot 1 && \text{thm } a+1=1 \\
&= a && \text{Identity}
\end{aligned}
$$

thm $\forall a,b \in \beta \quad a+\bar{a}\cdot b = a+b \qquad$ Dual $a\cdot(\bar{a}+b)=a\cdot b$

$$
\begin{aligned}
a+\bar{a}\cdot b &= (a+\bar{a})\cdot(a+b) && \text{Distributive} \\
&= 1\cdot(a+b) && \text{Complement} \\
&= a+b && \text{Identity}
\end{aligned}
$$

Boolean Functions:

Consider the boolean algebra
$(\{T, F\}, \vee, \wedge, \neg, T, F)$ but use the notation
$(\{1, 0\}, +, \cdot, -, 1, 0)$ $(\mathcal{B} = \{1, 0\})$

<u>Def</u> A boolean function of n variables is a function $f: \mathcal{B}^n \to \mathcal{B}$ where $\mathcal{B} = \{1, 0\}$

$$\mathcal{B}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathcal{B},\ 1 \le i \le n\}$$

<u>Ex</u> $f: \mathcal{B}^3 \to \mathcal{B}$ $f(x, y, z) = x \cdot y + \overline{z}$

x	y	z	f
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

Different boolean expressions determine the same boolean functions:

$$f(x_1, x_2, x_3) = x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$$
$$f(x_1, x_2) = \overline{x_1 \cdot x_2} = \overline{x_1} + \overline{x_2}$$

Standardized Form for boolean functions

<u>Def</u> A literal is a boolean variable or its complement: $y = x$ or $y = \overline{x}$

A minterm of the variables $x_1, x_2, \ldots, x_n$ is a boolean product of literals: $y_1 \cdot y_2 \cdot \cdots \cdot y_n$

Ex Two variables x_1, x_2

$y_1 \cdot y_2$	y_1 literal for x_1 y_2 literal for x_2
$x_1 \cdot x_2$	
$x_1 \cdot \overline{x_2}$	4 minterms for 2 variables
$\overline{x_1} \cdot \overline{x_2}$	2^n minterms for n variables
$\overline{x_1} \cdot x_2$	

there is a bijection between n tuples and minterms of n variables

$$(j_1, j_2, \dots, j_n) \longrightarrow y_1 \cdot y_2 \cdot \cdots \cdot y_n$$

If $j_i = 1$ then $y_i = x_i$

$j_i = 0$ then $y_i = \overline{x_i}$

Ex. $(1,0,0,1) \rightarrow x_1 \cdot \overline{x_2} \cdot \overline{x_3} \cdot x_4$

$(0,0,1,1,0) \longrightarrow \overline{x_1} \cdot \overline{x_2} \cdot x_3 \cdot x_4 \cdot \overline{x_5}$

Note: $\overline{x_1} \cdot \overline{x_2} \cdot x_3 \cdot x_4 \cdot \overline{x_5} = 1$ iff $x_1 = 0$, $x_2 = 0$, $x_3 = x_4 = 1$, $x_5 = 0$

Def Standardized form for boolean functions is called Sum of Products Form or Disjunctive Normal Form (DNF) and is the representation of the function as a sum of minterms.
(unique up to the ordering of minterms)

Ex. Find the sum of products form for

$$f(x,y,z) = (x+y) \cdot \overline{z}$$

① Calculate the truth table

x	y	z	$(x+y)\cdot\overline{z}$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

② Identify the n- tuples at which the function is 1

③ Form the minterms corresponding to these n-tuples and sum.

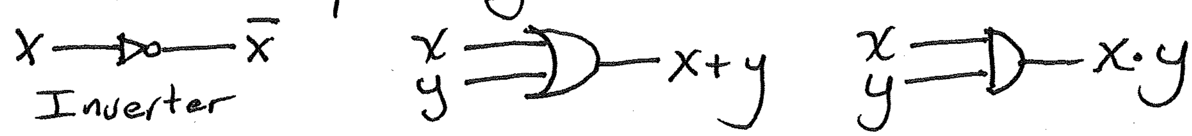
$$f(x,y,z) = x\cdot y\cdot\overline{z} + x\cdot\overline{y}\,\overline{z} + \overline{x}\cdot y\cdot\overline{z}$$

Sum of products form (DNF)

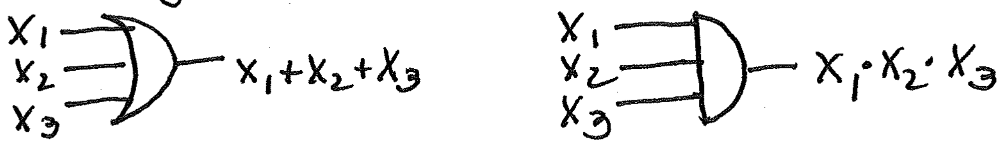
Note: We have also proved that every boolean function can be accomplished using disjunction, conjunction and negation.

Circuits - physical application of boolean functions

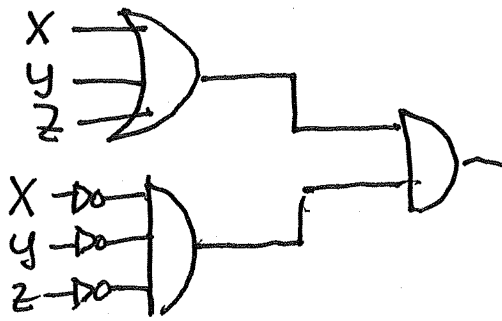
Circuits made up of gates:



Allow in design:



Represent a circuit with a boolean function:



$$f(x,y,z) = (x+y+z) \cdot (\bar{x} \cdot \bar{y} \cdot \bar{z})$$

Minimization of Circuits

Represent a circuit by a boolean function. Get an equivalent boolean function with a minimal number of terms in a sum of product expression.

EX. (Algebraic method)

$$f(x_1,x_2,x_3) = \bar{x}_1 \cdot x_2 \cdot \bar{x}_3 + \bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3 + x_1 \cdot \bar{x}_2 \cdot \bar{x}_3 + x_1 \cdot x_2 \cdot x_3 + x_1 \cdot \bar{x}_2 \cdot x_3$$

$$= \bar{x}_1 \cdot \bar{x}_3 (x_2 + \bar{x}_2) + x_1 \cdot \bar{x}_2 (\bar{x}_3 + x_3) + x_1 x_3 (x_2 + \bar{x}_2)$$

$$= \bar{x}_1 \cdot \bar{x}_3 (1) + x_1 \cdot \bar{x}_2 (1) + x_1 \cdot x_3 (1)$$

$$= \bar{x}_1 \cdot \bar{x}_3 + x_1 \bar{x}_2 + x_1 \cdot x_3$$

Karnaugh maps – visual aid to minimize circuits.

	y	$\bar{y}$
x		
$\bar{x}$		

Every cell represents a minterm. If the minterm appears in the sum of products form, put a 1 in the box. Adjacent cells represent minterms which differ in only one variable, so adjacent cells can be collapsed.

Combine 2 adjacent cells – eliminate 1 variable
4 adjacent cells – eliminate 2 variables
8 adjacent cells – eliminate 3 variables

Examples (Taken circuits, represented them by boolean functions, put in sum of products form, then filled in the Karnaugh map)

	y	$\bar{y}$
x	1	
$\bar{x}$	1	

$f(x,y) = y$

	y	$\bar{y}$
x		1
$\bar{x}$	1	1

$f(x,y) = \bar{x} + \bar{y}$

three variables:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	*			*
$\bar{x}$	**			**

(Right adjacent to left, *, **)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1			
$\bar{x}$	1			

$f(x,y,z) = y \cdot z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$	1	1		

$f(x,y,z) = y$ (rather than $yz + y\bar{z}$)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$				1

$f(x,y,z) = x + \bar{y}\cdot z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		1
$\bar{x}$	1	1		1

$f(x,y,z) = y + z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$			1	1

$f(x,y,z) = xy + \bar{x}\cdot\bar{y}$

Four variables

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	*			*
$w\bar{x}$				
$\bar{w}\bar{x}$				
$\bar{w}x$	*			*

* adjacent cells

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1	1		1
$w\bar{x}$	1	1		
$\bar{w}\bar{x}$			1	1
$\bar{w}x$				

$f(x,y,z,w) = wy + wxz + \bar{w}\bar{x}\bar{y}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1			1
$w\bar{x}$		1	1	
$\bar{w}\bar{x}$		1	1	
$\bar{w}x$	1			1

$f(x,y,z,w) = xz + \bar{x}\cdot\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx				1
$w\bar{x}$	1			1
$\bar{w}\bar{x}$	1	1	1	1
$\bar{w}x$	1	1	1	1

$f(x,y,z,w) = \bar{w} + z$