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ON A THEOREM OF BAERNSTEIN

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Abstract. In the paper *"A counterexample concerning integrability of derivatives of conformal mappings"* (Journal D'Analyse Mathematique, vol. 53 (1989)), Baernstein constructs Ω a simply connected domain in the plane for which the conformal mapping $f\colon \Omega \longmapsto \Delta$ (unit disc) satisfies

$$\int_{R\cap\Omega} |f'(z)|^p |dz| = \infty,$$

for some $p \in (1,2)$, where R is the real line.

This gives a counterexample to a well known conjecture stating that all the above integrals were finite for any $1 < p < 2$.

In the above paper everything reduces to prove a certain basic estimate. Baernstein, on the basis of numerical evidence provided by Donald Marshall, gives a proof of the theorem which consists in checking the numerical computation using Calculus, and asks for a conceptual proof of this basic estimate.

In this paper we present such a proof of Baernstein's theorem. The main tool in our proof is the method of the extremal metric.

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§1. Introduction.

Let us consider the following problem. Let Ω be a simply connected domain and $f\colon\Omega \longmapsto \delta$ (unit disc), be a conformal mapping. Assume L is a straight line which intersects the domain Ω, Hayman and Wu [4] showed that for any configuration as above,

$$\int_{L\cap\Omega} |f'(z)||dz| \leq C,$$

where C is an universal constant. Later Garnett, Gehring and Jones [3] simplified Hayman and Wu's proof and gave an improved value for the constant C. Fernandez, Heinonen and Martio in [2] give another proof of the same result with a better constant $C = 4\pi^2$, and a conjecture is offered for the best constant. In the same paper they showed that there exists a positive number p between 1 and 2, such that

$$\int_{L\cap\Omega} |f'(z)|^p|dz| \leq C,$$

with C and p constants independent of the configuration. It is not difficult to see that the line L may be taken to be the real axis R. The question is then for which exponents p is it true that $f'(z) \in L^P(R\cap\Omega)$, for any f and Ω? Taking Ω to be $\Delta\setminus(-1,0]$ one sees that $f'(z) \in L^2(R\cap\Omega)$ can fail. Baernstein conjectured that $f'(z) \in L^p(R\cap\Omega)$ would be true for any $1 < p < 2$.

Baernstein in [1] showed that this conjecture is not true. He constructed a simply connected domain Ω such that if we consider the conformal mapping f from Ω onto the unit disc, there exists a positive value p between 1 and 2, such that,

$$\int_{R\cap\Omega} |f'(z)|^p|dz| = \infty.$$

We pass to describe summarily the work done by Baernstein in [1]. His domain Ω is the complement of an infite tree T clustering to the real line, as in the picture below.

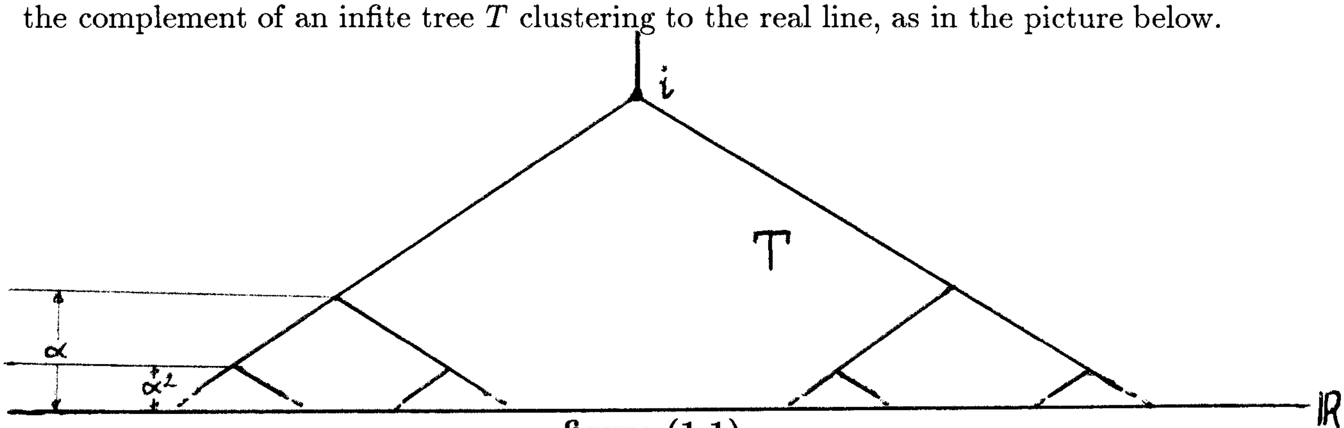

figure (1.1)

The fixed aperture at every branching of the tree T is $\frac{\pi}{3}$. Using Green functions, and the theory of conformal and quasiconformal mappings in a beautiful manner, Baernstein reduces everything to prove the following result. We need some preliminaries. Let us consider the domain $\Theta = C \setminus (-\infty, 1] \cup (0, e^{i\pi/3}]$, where $(0, e^{i\pi/3}]$ is the segment joining these two points. We are going to call $a = e^{i\pi/3}$, and consider the conformal mappings $F_i(z)$, $i = 1, 2$; mapping Θ onto the domain $H = C \setminus (-\infty, 0]$, such that $F_1(1) = 0$, $F_2(a) = 0$ and $|F_i(z)| \sim |z|$ as $z \to \infty$, $i = 1, 2$.

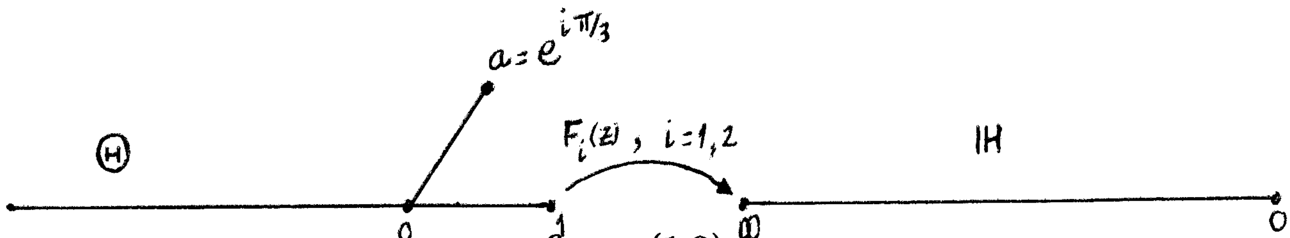

figure (1.2)

If we consider,

$$\gamma = \lim_{z \to 1} |\frac{F_1(z)}{z-1}|, \quad \beta = \lim_{z \to a} |\frac{F_2(z)}{z-a}|,$$

then Baernstein's theorem states that.

Theorem.

$$\gamma^{1/2} + \beta^{1/2} > \sqrt{2}.$$

In his paper Baernstein proves this result after numerical evidence given to him by Donald Marshall, who computed the values of γ and β using Trefethen's program [7], [5,p.422] for finding parameters for Schwarz- Christoffel transformations. He starts with the 4-place decimal approximation to the parameters given by the computer and confirm by Calculus the validity of the theorem, and asks for a conceptual proof of his theorem.

In this paper we present such a conceptual proof. In it our main tool is the method of the extremal metric. The idea of how to obtain lower bounds for γ and β using extremal metric was inspired to us by the paper of Jenkins and Oikawa [6], in which they obtain a sharp version of Ahlfors' distortion theorem, and then they use it to give simpler proofs of some well known results of Hayman.

Acknowledgments. I would like to thank Professor Albert Baernstein II for his helpful comments and suggestions concerning this work.

S2. Proof of the Theorem.

A. Estimating $\gamma = |\mathbf{F}_1'(\mathbf{1})|$.

Let ρ be a small positive number and consider the discs $D_\rho^{(1)} = \{z : |z-1| < \rho\}$, and $D_{1/\rho}^{(1)} = \{z : |z-1| < \frac{1}{\rho}\}$. Let $\Theta_\rho^{(1)}$ be the doubly connected domain

$$\Theta_\rho^{(1)} = \{[\Theta \cap D_{1/\rho}^{(1)}] \setminus \bar{D}_\rho^{(1)}\}$$

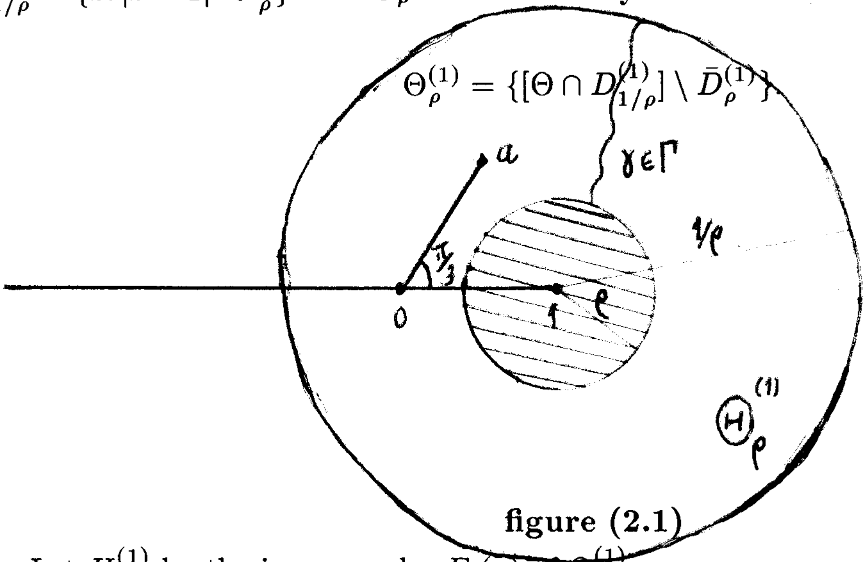

figure (2.1)

Let $H_\rho^{(1)}$ be the image under $F_1(z)$ of $\Theta_\rho^{(1)}$,

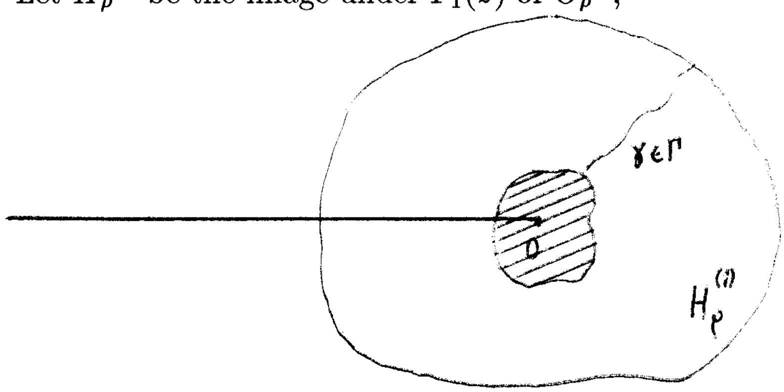

figure (2.2)

By the normalization properties of the function $F_1(z)$, it is not difficult to show that for any positive ϵ, there exists a small $\rho(\epsilon)$ positive such that,

$$\{z : |z| < \frac{1}{\rho(\epsilon)}(1-\epsilon)\} \subset F_1(D_{1/\rho(\epsilon)}^{(1)}) \subset \{z : |z| < \frac{1}{\rho(\epsilon)}(1+\epsilon)\}.$$

and

$$\{z : |z| < |F_1'(1)|(\rho(\epsilon)-\epsilon)\} \subset F_1(D_{\rho(\epsilon)}^{(1)}) \subset \{z : |z| < |F_1'(1)|(\rho(\epsilon)+\epsilon)\}.$$

Thus, if we consider in $\Theta_{\rho(\epsilon)}^{(1)}$ the module problem for the family of curves joining $\partial D_{\rho(\epsilon)}^{(1)}$ with $\partial D_{1/\rho(\epsilon)}^{(1)}$ using the conformal invariance of the module and the comparison property for the modules, we have that

$$mod_{\Theta^{(1)}_{\rho(\epsilon)}}(\Gamma) \leq \frac{2\pi}{\ln(\frac{(1-\epsilon)}{\rho(\epsilon)(\rho(\epsilon)+\epsilon)|F_1'(1)|})}.$$

This provides with a lower bound for the module, our goal is to obtain an upper bound for the same module. For this we consider the conformal mapping, $\Phi(z) = \ln(z-1)$, then

$$\Phi(z)\colon \Theta^{(1)}_{\rho(\epsilon)} \longmapsto S^{(1)}_{\rho(\epsilon)},$$

where $S^{(1)}_{\rho(\epsilon)}$ is a quadrangle.

$\tilde{\Gamma}$ is the family of curves in $S^{(1)}_{\rho(\epsilon)}$ joining the pair of sides opposite to the vertical sides. By the conformal invariance of the module we have the following equality

$$mod_{\Theta^{(1)}_{\rho(\epsilon)}}(\Gamma) = mod_{S^{(1)}_{\rho(\epsilon)}}(\bar{\Gamma}),$$

where $\bar{\Gamma}$ is tha family of curves in $S^{(1)}_{\rho(\epsilon)}$ joining the pair of vertical sides. By a well known property of the modules of the families $\tilde{\Gamma}$ and $\bar{\Gamma}$, we have that

$$mod_{S^{(1)}_{\rho(\epsilon)}}(\bar{\Gamma}) = \frac{1}{mod_{S^{(1)}_{\rho(\epsilon)}}(\tilde{\Gamma})}.$$

Thus, to obtain a lower bound for $mod_{\Theta^{(1)}_{\rho(\epsilon)}}(\Gamma)$, we need an upper bound of $mod_{S^{(1)}_{\rho(\epsilon)}}(\tilde{\Gamma})$.

The idea of how to obtain the right upper bound of $mod_{S^{(1)}_{\rho(\epsilon)}}(\tilde{\Gamma})$ was suggested by [7]. For any value of x in the interval $\ln\rho(\epsilon) < x < -\ln\rho(\epsilon)$, let $\sigma(x)$ denote the maximal open subinterval of $Re\{z\} = x$ in $S^{(1)}_{\rho(\epsilon)}$ such that the two components of $S^{(1)}_{\rho(\epsilon)} \setminus \sigma(x)$ have the two vertical sides as boundary components. Let $\theta(x)$ denote the length of $\sigma(x)$ and $\theta_1(x)$ the length of the part of the segment $\sigma(x)$ below the x-axis., $\theta_2(x)$ the length of the part above the x-axis. As it can be easily seen, $\theta_1(x) = \pi$ for any x in the interval $\ln\rho(\epsilon) < x < -\ln\rho(\epsilon)$; for $\theta_2(x)$ we have

$$\theta_2(x) = \begin{cases} \pi, & \text{if } \ln\rho(\epsilon) < x < \ln(\frac{\sqrt{3}}{2}); \\ \tilde{\theta}_2(x), & \text{if } \ln(\frac{\sqrt{3}}{2}) < x < 0; \\ \pi, & \text{if } 0 \leq x < -\ln\rho(\epsilon). \end{cases}$$

Let the interval $(\ln(\frac{\sqrt{3}}{2}), 0)$ be divided into n consecutive closed subintervals Δ_j, $j = 1, \ldots, n$, of equal length, and for each $j = 1, \ldots, n$, let

$$\theta^{(s)}_{2,j}(x) = \min_{t \in \Delta_j} \theta_2(t),$$

and define $\theta_2^{(s)}(x) = \theta_{2,j}^{(s)}(x)$ if $x \in \Delta_j$; $j = 1, \ldots, n$. It is clear that such minimum is attained. At the end point $\bar{x}$ of an interval Δ_j the step function $\theta_2^{(s)}(x)$ has a negative jump, then we draw the ray given by $\bar{x} - \lambda$, $\theta_{2,j}^{(s)}(\bar{x}) + \lambda$; $\lambda \geq 0$; $j = 1 \ldots, n$. The lower envelope of these rays and the locus $y = \theta_2^{(s)}(x)$ defines on the interval $(\ln(\frac{\sqrt{3}}{2}), 0)$ a continuous function $\theta_2^{(t)}(x)$, which determines a decomposition of the interval into a finite number of subintervals on which the locus $y = \theta_2^{(t)}(x)$ has slope -1 or 0. We define $\theta_2^{(t)}(x)$ in the interval $(\ln \rho(\epsilon), -\ln \rho(\epsilon))$ by

$$\theta_2^{(t)}(x) = \begin{cases} \pi, & \text{if } \ln \rho(\epsilon) < x \leq \ln(\frac{\sqrt{3}}{2}) + \Delta_1 - 2\pi + \theta_{2,2}^{(s)}; \\ \theta_{2,2}^{(s)} - (x - \ln(\frac{\sqrt{3}}{2}) - \Delta_2), & \text{if } \ln(\frac{\sqrt{3}}{2}) + \Delta_1 - 2\pi + \theta_{2,2}^{(s)} < x < \ln(\frac{\sqrt{3}}{2}) + \Delta_1; \\ \theta_2^{(t)}(x), & \text{if } \Delta_1 + \ln(\frac{\sqrt{3}}{2}) < x < 0; \\ \frac{2\pi}{3} + \lambda x^2, & \text{if } 0 \leq x < \sqrt{\frac{\pi}{3\lambda}}; \\ \pi, & \text{if } \sqrt{\frac{\pi}{3\lambda}} \leq x < -\ln \rho(\epsilon). \end{cases}$$

The domain determined by

$$-\theta_1(x) < y < \theta_2^{(t)}(x), \quad \ln \rho(\epsilon) < x < -\ln \rho(\epsilon),$$

becomes a quadrangle $Q_{\rho(\epsilon)}^{(1)}$ (figure (2.4)) on assigning as a pair of opposite sides the segments

$$-\theta_1(\ln \rho(\epsilon)) < y < \theta_2^{(t)}(\ln \rho(\epsilon))$$

and,

$$-\theta_1(-\ln \rho(\epsilon)) < y < \theta_2^{(t)}(-\ln \rho(\epsilon)).$$

For the module in $Q_{\rho(\epsilon)}^{(1)}$ of the family of curves joining the pair of sides complementary to the two vertical sides, we have that

$$mod_{S_{\rho(\epsilon)}^{(1)}}(\tilde{\Gamma}) \leq mod_{Q_{\rho(\epsilon)}^{(1)}}(\tilde{\Gamma}).$$

Thus it is enough to obtain an upper bound for $mod_{Q_{\rho(\epsilon)}^{(1)}}(\tilde{\Gamma})$. It is known that an upper bound for this module is given by the Dirichlet integral of a piecewise continuously differentiable function in $Q_{\rho(\epsilon)}^{(1)}$ taking the value 0 on the side given by $y = -\theta_1(x)$, and the value 1 on the side given by $y = \theta_2^{(t)}(x)$. Such a function is given by

$$u(x,y) = \frac{y + \theta_1(x)}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x) = \theta_1(x) + \theta_2^{(t)}(x)$. To estimate the Dirichlet integral of $u(x,y)$ we subdivide the domain $Q^{(1)}_{\rho(\epsilon)}$ into five pieces each corresponding to one of the following intervals in the x-axis; $I = (\ln\rho(\epsilon),\ \ln(\frac{\sqrt{3}}{2}) + \Delta_1 - 2\pi + \theta^{(s)}_{2,2})$; $II = (\ln(\frac{\sqrt{3}}{2}) - 2\pi + \Delta_1 + \theta^{(s)}_{2,2},\ \ln(\frac{\sqrt{3}}{2}) + \Delta_1)$; $III = (\ln(\frac{\sqrt{3}}{2}) + \Delta_1,\ 0)$; $IV = (0,\ \sqrt{\frac{\pi}{3\lambda}})$ and $V = (\sqrt{\frac{\pi}{3\lambda}},\ -\ln\rho(\epsilon))$.

On the two pieces of the Dirichlet integral corresponding to the intervals I and V $u(x,y) = \frac{y+\pi}{2\pi}$, and since when we take the limit as the number of subdivisions $n \to \infty$ then $\Delta_1 \to 0$ and $\theta^{(s)}_{2,2} \to \frac{5\pi}{6}$, therefore

$$\iint_I + \iint_V |\nabla u(x,y)|^2 dx\ dy = \frac{1}{2\pi}\ln(\frac{1}{\rho(\epsilon)}) + \frac{1}{2\pi}[\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}] + \frac{1}{2\pi}\ln(\frac{1}{\rho(\epsilon)}) - \frac{1}{2\pi}\sqrt{\frac{\pi}{3\lambda}}.$$

It is not difficult to see that the Dirichlet integral corresponding to II after we let $n \to \infty$ tends to

$$\int_{\ln(\frac{\sqrt{3}}{2})-\pi/6}^{\ln(\frac{\sqrt{3}}{2})} \int_{-\pi}^{\frac{5\pi}{6}-(x-\ln(\frac{\sqrt{3}}{2}))} |\nabla(\frac{y+\pi}{\frac{11\pi}{6} - (x - \ln(\frac{\sqrt{3}}{2}))})|^2 dx\ dy =$$

$$= \int_{\ln(\frac{\sqrt{3}}{2})-\pi/6}^{\ln(\frac{\sqrt{3}}{2})} \int_{-\pi}^{\frac{5\pi}{6}-(x-\ln(\frac{\sqrt{3}}{2}))} [\frac{1}{(\frac{11\pi}{6} - (x - \ln(\frac{\sqrt{3}}{2})))^2} + \frac{(y+\pi)^2}{(\frac{11\pi}{6} - (x - \ln(\frac{\sqrt{3}}{2})))^4}] dx\ dy$$

$$= \int_{\ln(\frac{\sqrt{3}}{2})-\pi/6}^{\ln(\frac{\sqrt{3}}{2})} \frac{4}{3}[\frac{1}{(\frac{11\pi}{6} - (x - \ln(\frac{\sqrt{3}}{2})))}] dx$$

$$= \frac{4}{3}[-\ln(\frac{11\pi}{6} - (x - \ln(\frac{\sqrt{3}}{2})))]_{\ln(\frac{\sqrt{3}}{2})-\pi/6}^{\ln(\frac{\sqrt{3}}{2})}$$

$$= \frac{4}{3}\ln\frac{12}{11}.$$

As for the piece corresponding to IV, we have that after some calculations

$$\iint_{IV} |\nabla(\frac{y+\pi}{\frac{5\pi}{3} + \lambda x^2})|^2 dx\ dy$$

$$= [\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}} - \frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}] \arctan(\frac{1}{5}) + \frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda}.$$

The estimate corresponding to III is more delicate, and we will treat it carefully.

$$\iint_{III} |\nabla u(x,y)|^2 dx\ dy = \int_{\ln(\frac{\sqrt{3}}{2})}^{0} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_{\ln(\frac{\sqrt{3}}{2})}^{0} \frac{\theta_1'(x)^2 - \theta_2^{(t)\prime}(x)\theta_1'(x) + \theta_1'(x)^2}{\theta^{(t)}(x)} dx$$

$$= \int_{\ln(\frac{\sqrt{3}}{2})}^{0} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\sum_{j=1}^{n}\int_{\Delta_j} \frac{1}{\theta^{(t)}(x)} dx = (i) + (ii).$$

We proceed to estimate these two integrals.

$$(i) = \int_{\ln(\frac{\sqrt{3}}{2})}^{0} \frac{dx}{\theta^{(t)}(x)} \leq -\frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2}).$$

We estimate (ii) as follows,

$$(ii) = \frac{1}{3}\sum_{j=1}^{n}\int_{\Delta_j} \frac{dx}{\theta^{(t)}(x)} = \frac{1}{3}\sum_{j=1}^{n}\int_{\Delta_j} \frac{-1}{\pi + \theta_j^{(s)} - x} dx$$

$$= \sum_{j=1}^{n} \frac{1}{3}\left[\ln(\pi + \theta_j^{(s)} - x)\right]_0^{V_j} \to \frac{1}{3}\ln\frac{11}{10},$$

as $n \to \infty$, where V_j is the total variation of the function $\theta_2^{(t)}(x)$ in the interval Δ_j, $j = 1, \ldots, n$. This completes our estimates. Putting all of them together we obtain that

$$mod_{Q^{(1)}_{\rho(\epsilon)}}(\tilde{\Gamma}) \leq \frac{1}{2\pi}\ln(\frac{1}{\rho^2(\epsilon)}) + \frac{1}{2\pi}[\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}] - \frac{1}{2\pi}\sqrt{\frac{\pi}{3\lambda}} + \frac{4}{3}\ln\frac{12}{11} - \ln(\frac{\sqrt{3}}{2})\frac{3}{5\pi}$$

$$+[\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}} - \frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}]\arctan(\frac{1}{5}) + \frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda} + \frac{1}{200} + \frac{1}{3}\ln\frac{11}{10}.$$

Let us call $G(\lambda)$ the expression on the right hand side of the above inequality involving λ,

$$G(\lambda) = [\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}} - \frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}]\arctan(\frac{1}{5}) + \frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda} - \frac{1}{2\pi}\sqrt{\frac{\pi}{3}}\frac{1}{\sqrt{\lambda}},$$

and solve the equation $G(\lambda) = 0$.

$$[\frac{4}{3}\sqrt{\frac{\pi}{3}} - \frac{4}{3}\sqrt{\frac{5\pi}{3}}\arctan(\frac{1}{5})]\sqrt{\lambda} = [\frac{1}{2\pi}\sqrt{\frac{\pi}{3}} - \sqrt{\frac{3}{5\pi}}\arctan(\frac{1}{5})]\frac{1}{\sqrt{\lambda}},$$

thus,

$$\lambda = \frac{\frac{1}{2\pi}\sqrt{\frac{\pi}{3}} - \sqrt{\frac{3}{5\pi}}\arctan(\frac{1}{5})}{\frac{4}{3}\sqrt{\frac{\pi}{3}} - \frac{4}{3}\sqrt{\frac{5\pi}{3}}\arctan(\frac{1}{5})} = 0.10050259.$$

Choosing λ to be this value the expression on the right hand side of the inequality involving λ is equal to zero, therefore;

$$mod_{Q^{(1)}_{\rho(\epsilon)}}(\tilde{\Gamma}) \leq \frac{1}{2\pi}\ln(\frac{1}{\rho^2(\epsilon)}) + \frac{1}{2\pi}(\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}) + \frac{1}{200} + \frac{1}{3}\ln\frac{11}{10} + \frac{4}{3}\ln\frac{12}{11} - \frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2}).$$

Thus,

$$\frac{2\pi}{\ln(\frac{(1-\epsilon)}{\rho(\epsilon)(\rho(\epsilon)+\epsilon)|F_1'(1)|})} \geq$$

$$\geq \frac{1}{\frac{1}{2\pi}\ln(\frac{1}{\rho^2(\epsilon)}) + \frac{1}{2\pi}(\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}) + \frac{1}{200} + \frac{1}{3}\ln\frac{11}{10} + \frac{4}{3}\ln\frac{12}{11} - \frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2})};$$

taking inverses and exponentiating both sides, we obtain

$$\frac{1}{\rho^2(\epsilon)}e^{\{(\ln(\frac{\sqrt{3}}{2})-\frac{\pi}{6})+2\pi(\frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2}))\}} \geq$$

$$\geq \frac{1-\epsilon}{(\rho(\epsilon)+\epsilon)\rho(\epsilon)|F_1'(1)|}.$$

Letting $\epsilon \to 0$ we get that,

$$\gamma = |F_1'(1)| \geq$$

$$\geq \frac{1}{e^{\{(\ln(\frac{\sqrt{3}}{2})-\frac{\pi}{6})+2\pi(\frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2}))\}}} = \alpha.$$

Hence,

$$\gamma^{1/2} = |F_1'(1)|^{1/2} \geq \alpha^{1/2} \geq 0.79249.$$

B. Estimating $\beta = |\mathbf{F}_2'(\mathbf{a})|$.

Let ρ be a small positive number, and consider the discs $D_\rho^{(2)} = \{z\colon |z-a| < \rho\}$, and $D_{1/\rho}^{(2)} = \{z\colon |z-a| < \frac{1}{\rho}\}$. Let $\Theta_\rho^{(2)}$ be the doubly connected domain

$$\Theta_\rho^{(2)} = \{[\Theta \cap D_{1/\rho}^{(2)}] \setminus \bar{D}_\rho^{(2)}\}.$$

Let $H_\rho^{(2)}$ be the image under $F_2(z)$ of $\Theta_\rho^{(2)}$.

By the same reason than in the first estimate **A**, for any positive ϵ, there exists a small $\rho(\epsilon)$ positive such that;

$$\{z\colon |z| < |F_2'(a)|(\rho(\epsilon)-\epsilon)\} \subset F_2(D_{\rho(\epsilon)}^{(2)}) \subset \{z\colon |z| < |F_2'(a)|(\rho(\epsilon)+\epsilon)\}$$

and,

$$\{z\colon |z| < \frac{1}{\rho(\epsilon)}(1-\epsilon)\} \subset F_2(D_{1/\rho(\epsilon)}^{(2)}) \subset \{z\colon < \frac{1}{\rho(\epsilon)}(1+\epsilon)\}.$$

Considering the analogous problem in $\Theta_{\rho(\epsilon)}^{(2)}$, we have that;

$$mod_{\Theta_{\rho(\epsilon)}^{(2)}} \leq \frac{2\pi}{\ln(\frac{(1-\epsilon)}{\rho(\epsilon)(\rho(\epsilon)+\epsilon)|F_2'(a)|})}.$$

We want to obtain an upper bound for the module above. For this we consider the conformal mapping $\Psi(z) = \ln(z-a)$,

$$\Psi(z)\colon \Theta_{\rho(\epsilon)}^{(2)} \longmapsto S_{\rho(\epsilon)}^{(2)},$$

where $S_{\rho(\epsilon)}^{(2)}$ is again a quadrangle.

Where $\tilde{\Gamma}$ is the family of curves in $S_{\rho(\epsilon)}^{(2)}$ joining the pair of sides opposite to the vertical sides of $S_{\rho(\epsilon)}^{(2)}$. By the conformal invariance of the module we have that,

$$mod_{\Theta_{\rho(\epsilon)}^{(2)}}(\Gamma) = mod_{S_{\rho(\epsilon)}^{(2)}}(\bar{\Gamma}),$$

where $\bar{\Gamma}$ is the family of curves in $S_{\rho(\epsilon)}^{(2)}$ joining the pair of vertical sides. By a well known property of the modules,

$$mod_{S_{\rho(\epsilon)}^{(2)}}(\bar{\Gamma}) = \frac{1}{mod_{S_{\rho(\epsilon)}^{(2)}}(\tilde{\Gamma})}.$$

Thus, to obtain a lower bound for $mod_{\Theta^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma})$, we need an upper bound for the module $mod_{S^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma})$. To obtain our bound on the left hand side of $S^{(2)}_{\rho(\epsilon)}$, i.e. $\{z\colon z \in S^{(2)}_{\rho(\epsilon)};\ Re\{z\} \leq 0\}$, we proceed as in case **A**. Our function $\theta_2(x)$ in this case is given by,

$$\theta_2(x) = \begin{cases} \frac{4\pi}{3}, & \text{if } \ln\rho(\epsilon) < x \leq 0; \\ \pi + \arctan(\sqrt{\frac{3}{4e^{2x}-3}}), & \text{if } 0 \leq x < -\ln\rho(\epsilon). \end{cases}$$

We modify the function $\theta_1(x)$ in the same way we did with $\theta_2(x)$ in case **A** for values of $\ln\rho(\epsilon) < 0 \leq 0$, and to the right of $x = 0$ we are going to modify $\theta_1(x)$ as follows;

$$\theta_1^{(t)}(x) = \begin{cases} \frac{\pi}{3} + \delta x, & \text{if } 0 \leq x < \lambda; \\ \theta_1(x), & \text{if } \lambda \leq x \leq -\ln\rho(\epsilon). \end{cases}$$

Where $\delta > 0$ is a free parameter and λ is implicitely defined by the equation

$$\delta\lambda + \arctan(\sqrt{\frac{3}{4e^{2x}-3}}) = \frac{2\pi}{3}.$$

The domain determined by

$$-\theta_1^{(t)}(x) < y < \theta_2(x);\ \ \ln\rho(\epsilon) < x < -\ln\rho(\epsilon),$$

becomes a quadrangle $Q^{(2)}_{\rho(\epsilon)}$ on assigning as a pair of opposite sides the segments

$$-\theta_1^{(t)}(\ln\rho(\epsilon)) < y < \theta_2(\ln\rho(\epsilon))$$

and

$$-\theta_1^{(t)}(-\ln\rho(\epsilon)) < y < \theta_2(-\ln\rho(\epsilon)).$$

As in the case **A** we have that;

$$mod_{S^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma}) \leq mod_{Q^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma}).$$

Thus, it is enough to obtain an upper bound for the module $mod_{Q^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma})$. This upper bound is given by the Dirichlet integral of the function

$$u(x,y) = \frac{\theta_2(x) - y}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2(x)$. Hence,

$$\iint_{Q^{(2)}_{\rho(\epsilon)}} |\nabla u(x,y)|^2 dx\ dy =$$

$$= \iint_{Q^{(2)}_{\rho(\epsilon)} \cap \{Re\{z\} \leq 0\}} + \iint_{Q^{(2)}_{\rho(\epsilon)} \cap \{Re\{z\} > 0\}} |\nabla u(x,y)|^2 dx\ dy = I + II.$$

The estimate of the integral I is the same as in case **A** because if we look at the left hand sides of the domains $Q^{(1)}_{\rho(\epsilon)}$ and $Q^{(2)}_{\rho(\epsilon)}$, they are the same up to a vertical translation. Thus,

$$I \leq \frac{1}{2\pi \ln(\frac{1}{\rho(\epsilon)}) + \frac{1}{3}\ln\frac{11}{10} + \frac{1}{200} + \frac{1}{2\pi}\left[\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}\right] + \frac{4}{3}\ln\frac{12}{11} - \frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2})}.$$

We pass to estimate the second integral II.

$$II = \iint_{Q^{(2)}_{\rho(\epsilon)} \cap \{Re\{z\} > 0\}} |\nabla u(x,y)|^2 dx\ dy$$

$$= \int_0^{-\ln \rho(\epsilon)} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_0^{-\ln\rho(\epsilon)} \frac{\theta_1^{(t)\prime}(x)^2 - \theta_1^{(t)\prime}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)} dx,$$

where,

$$\theta_2(x) = \pi + \arctan(\sqrt{\frac{3}{4e^{2x} - 3}}),$$

for, $0 \leq x \leq -\ln \rho(\epsilon)$, and

$$\theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2(x) = 2\pi,$$

for values of x such that $\lambda \leq x \leq -\ln \rho(\epsilon)$. Thus II is equal to;

$$II = \int_0^{\lambda} \frac{1}{\theta^{(t)}(x)} dx + \frac{1}{3}\int_0^{\lambda} \frac{\theta_1^{(t)\prime}(x)^2 - \theta_1^{(t)\prime}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)} dx$$

$$+ \frac{1}{2\pi}\ln\frac{1}{\rho(\epsilon)} - \frac{\lambda}{2\pi} + \frac{1}{6\pi}\int_{\lambda}^{-\ln\rho(\epsilon)} 3(\theta_2'(x))^2 dx.$$

Let us compute the last integral in the above equality,

$$\int_{\lambda}^{-\ln\rho(\epsilon)} (\theta_2'(x))^2 dx = 3\int_{\lambda}^{-\ln\rho(\epsilon)} \frac{dx}{4e^{2x} - 3} = \frac{1}{2}\ln(\frac{4 - 3\rho^2(\epsilon)}{4 - 3e^{-2\lambda}}).$$

It remains to estimate:

$$\int_0^\lambda \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_0^\lambda \frac{\theta_1^{(t)\prime}(x)^2 - \theta_1^{(t)\prime}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)}dx$$

$$= \int_0^\lambda \frac{1 + \frac{\delta^2}{3} + \frac{1}{4e^{2x}-3} + \frac{\delta}{3}\sqrt{\frac{3}{4e^{2x}-3}}}{\frac{4\pi}{3} + \delta x + \arctan(\sqrt{\frac{3}{4e^{2x}-3}})}dx$$

$$= \int_0^\lambda \frac{1 + \frac{2\delta^2}{3} + \frac{1}{4e^{2x}-3}}{\frac{4\pi}{3} + \delta x + \arctan(\sqrt{\frac{3}{4e^{2x}-3}})}dx - \frac{\delta}{3}\big[\ln(\delta x + \frac{4\pi}{3} + \arctan(\sqrt{\frac{3}{4e^{2x}-3}}))\big]_0^\lambda.$$

The second term in the formula above is equal to $\frac{\delta}{3}\ln\frac{6}{5}$. Thus, to complete our estimate, our final goal is to find a suitable bound for the following integral.

$$\int_0^\lambda \frac{1 + \frac{2\delta^2}{3} + \frac{1}{4e^{2x}-3}}{\frac{4\pi}{3} + \delta x + \arctan(\sqrt{\frac{3}{4e^{2x}-3}})}dx - \frac{\lambda}{2\pi},$$

where $\delta\lambda + \arctan(\sqrt{\frac{3}{4e^{2\lambda}-3}}) = \frac{2\pi}{3}$, our first observation is that;

$$\frac{4\pi}{3} + \delta x + \arctan(\sqrt{\frac{3}{4e^{2x}-3}}) \geq 2\pi - \arctan(\sqrt{\frac{3}{4e^{2x}-3}}),$$

for $0 < x < \lambda$, therefore it is enough to estimate the integral

$$\int_0^\lambda \Big[\frac{1 + \frac{1}{4e^{2x}-3}}{2\pi - \arctan(\sqrt{\frac{3}{4e^{2x}-3}})} - \frac{1}{2\pi}\Big]dx$$

$$\leq \int_0^\infty \frac{\frac{2\pi}{4e^{2x}-3} + \arctan(\sqrt{\frac{3}{4e^{2x}-3}})}{2\pi(2\pi - \arctan(\sqrt{\frac{3}{4e^{2x}-3}}))}dx.$$

In the above integral we have dropped the term $\frac{2\delta^2}{3}$ in the numerator, since δ can be made as small as we please. Using the change of variable $u = \arctan(\sqrt{\frac{3}{4e^{2x}-3}})$, the above integral becomes;

$$-\int_{\frac{\pi}{3}}^0 \frac{\frac{2\pi}{3}(\tan u)^2 + u}{2\pi(2\pi - u)}\frac{du}{\tan u}$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{3}} \frac{2\pi}{3}\frac{\tan u}{(2\pi - u)}du + \frac{1}{2\pi}\int_0^{\frac{\pi}{3}} \frac{u}{(2\pi - u)}\cot u \;\; du = \mathbf{1} + \mathbf{2}.$$

Standard numerical integration methods give us the following estimates from above for the two integrals **1** and **2**.

$$\mathbf{1} \leq \frac{1}{3}(0.126) = 0.042,$$

and,

$$\mathbf{2} \leq \frac{1}{2\pi}(0.158) \leq 0.0252.$$

Putting all these estimates together we get that; after letting $\delta \to 0$, and $\lambda \to \infty$.

$$mod_{Q^{(2)}_{\rho(\epsilon)}}(\tilde{\Gamma}) \leq \frac{1}{2\pi}\ln\frac{1}{\rho^2(\epsilon)} + \frac{1}{3}\ln\frac{11}{10} + \frac{1}{200} + \frac{1}{2\pi}[\ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{6}]$$

$$+\frac{4}{3}\ln\frac{12}{11} - \frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2}) + \frac{1}{4\pi}\ln(\frac{4-3\rho^2(\epsilon)}{4}) + 0.042 + 0.0252.$$

Thus, taking inverses and exponentiating as we did in case **A**, we obtain that, after we let $\epsilon \to 0$:

$$\beta = |F_2'(a)| \geq \frac{1}{e^{2\pi\{\frac{1}{3}\ln\frac{11}{10}+\frac{1}{200}+\frac{1}{2\pi}(\ln(\frac{\sqrt{3}}{2})-\frac{\pi}{6})+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln(\frac{\sqrt{3}}{2})+0.0672\}}} = \eta.$$

Hence,

$$\beta^{1/2} = |F_2'(a)|^{1/2} \geq \eta^{1/2} \geq 0.6403.$$

Therefore putting together the two estimates, we have that;

$$|F_2'(a)|^{1/2} + |F_1'(1)|^{1/2} = \beta^{1/2} + \gamma^{1/2} > 0.79249 + 0.6403 > \sqrt{2},$$

as we wanted to show.

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