INTERPOLATION IN THE UNIT BALL OF $\mathbf{C}^n$

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Abstract: A necessary and sufficient condition is given for a discrete multiplicity variety in the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ to be an interpolating variety for weighted spaces of holomorphic functions in $\mathbf{B}_n$.

§1. **Introduction.** In this paper, we shall consider when a discrete multiplicity variety in the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ is an interpolating variety for holomorphic functions in $\mathbf{B}_n$ with growth conditions.

Let f be a holomorphic function in $\mathbf{B}_n$ and $\{\zeta_k\}$ a discrete set in $\mathbf{B}_n$. Then we have the following Taylor expansion about each ζ_k:

$$f(z) = \sum_{|I|=0}^{\infty} f_{k,I}(z-\zeta_k)^I,$$

where (and throughout the paper) $f_{k,I} := \frac{1}{I!}\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I}$, $I := (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, $\mathbf{Z}^+ = \{0, 1, 2\cdots\}$, and $|I| = i_1 + i_2 + \cdots + i_n$.

Let $\{m_k\}$ be a sequence of positive integers. We consider the following interpolation problem with multiplicities in the unit ball: under what (necessary and sufficient) conditions is it true that for any multi-indexed sequence $\{a_{k,I}\}_{k\in\mathbf{N}, 0\le|I|<m_k}$ of complex numbers satisfying a certain growth condition (defined in §2) there exists a holomorphic function f in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that

$$f_{k,I} = a_{k,I}, \quad \text{for} \quad k \in \mathbf{N}, 0 \le |I| < m_k, \tag{1.1}$$

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where $I := (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, and $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is the sapce of holomorphic functions in $\mathbf{B}_n$ satisfing that

$$(1 - |z|)^A |f(z)| < B, z \in \mathbf{B}_n$$

for some constants $A, B > 0$. We will then say that $V := \{(\zeta_k, m_k)\}$ is an interpolating (multiplicity) variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Note that the condition (1.1) means that f has a prescribed finite collection of Taylor coefficients at each ζ_k. In the special case that $m_k = 1$ for all k, (1.1) simply means that f takes prescribed values at each ζ_k.

The similar interpolation problem for weighted spaces of entire functions in $\mathbf{C}^n$ has been studied extensively due to its applications to other subjects such as harmonic analysis (see [BG], [BKS], [BL1], [BL2], [BT], [BV], [LT], [LV], [S], etc.). Given a discrete set $V = \{\zeta_k\}$ in $\mathbf{C}^n$, a necessary and sufficient interpolation condition in terms of the "directional derivatives" of defining functions was found in [BL1] for V to be an interpolating variety for the space $A_p(\mathbf{C}^n)$, the algebra of entire functions in $\mathbf{C}^n$ satisfying that $|f(z)| \leq A\exp(Bp(z))$ for some $A, B > 0$ in the sense of Berenstein and Taylor ([BT]), where p is a plurisubharmonic weight function in $\mathbf{C}^n$. It was showed in [M] that this condition can be carried over to $\mathbf{B}_n$ for a discrete set $\{\zeta_k\}$ in $\mathbf{B}_n$ to be interpolating for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. In this paper, we consider when an arbitrary given multiplicity variety $V = \{(\zeta_k, m_k)\}$ in the unit ball $\mathbf{B}_n$ is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. It seems hard to give an analytic interpolation condition in terms of "directional derivatives " of defining functions simiar to the one in [BL1] or [M]. The conditions obtained here are given using the distribution of points of V in the "tube"

$$S(F; \epsilon, C) := \{z \in \mathbf{B}_n : |F(z)| := (\sum_{j=1}^{n} |f_j(z)|^2)^{\frac{1}{2}} < \epsilon(1 - |z|)^C\}, \qquad (1.2)$$

where $F = (f_1, \cdots, f_m)$ is a defining vector function. It turns out that a multiplicity variety $V = \{(\zeta_k, m_k)\}$ in $\mathbf{B}_n$ is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if and only if there exist constants $\epsilon, C > 0$, and $m(\geq n)$ holomorphic functions $f_1, \cdots, f_m$ in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that these functions vanish at each ζ_k with multiplicity $\geq m_k$, and each component of the "tube" $S(F; \epsilon, C)$ defined as in (1.2), where $F = (f_1, \cdots, f_m)$, contains at most one point ζ_k and the diameter of such a component is at most $\lambda(1 - |\zeta_k|)^C$ for some suitable constant $0 < \lambda < 1$(see Theorem 2.7).

We refer the reader to [BV] for similar interpolation problems with multiplicities in weighted spaces of entire functions in $\mathbf{C}^n$.

§2. Prelimilaries and Results.

First of all, let us fix some notations, which will be used throughout the paper.

Definition 2.1. Let $\mathbf{H}(\mathbf{B}_n)$ be the ring of all holomorphic functions in $\mathbf{B}_n$. Then

$$\mathbf{A}^{-\infty}(\mathbf{B}_n) = \{f \in \mathbf{H}(\mathbf{B}_n) : \sup_{z \in \mathbf{B}_n} \frac{\log|f(z)|}{\log \frac{e}{1-|z|}} < \infty\}.$$

Note that it is not the specific growth conditions on the functions $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ which are important, but rather their consequences for the ring $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. The growth condition imposed on the holomorphic functions implies that $\mathbf{A}^{-\infty}(\mathbf{B}_n) \supset H^\infty(\mathbf{B}_n)$, the space of bounded holomorphic functions in $\mathbf{B}_n$, and that $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is closed under differentiation. The main theorem in the paper still holds if the space $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ is replaced by the space

$$A_p(\mathbf{B}_n) := \{f \in \mathbf{H}(\mathbf{B}_n) : |f(z)| \leq Ae^{Bp(\frac{1}{1-|z|})}$$

for some $A, B > 0\}$, where p is a proper increasing function so that the calculation in the proof of the paper can be carried out similarily. The space $\mathbf{A}^{-\infty}(\mathbf{B}_n)$

can be thought as the union of the weighted spaces

$$A^{-\alpha} := \{f \in \mathbf{H}(\mathbf{B}_n) : \sup_{z\in\mathbf{B}_n} (1-|z|)^\alpha |f(z)| < \infty\}, \alpha > 0$$

and the union of the weighted Bergman spaces

$$B_{\alpha,\beta} := \{f \in \mathbf{A}^{-\infty}(\mathbf{B}_n) : \int_{\mathbf{B}_n} (1-|z|)^\alpha |f(z)|^\beta dm(z) < \infty\}, \alpha > -1, \beta > 0.$$

It carries the natual topology as an inductive limit of Banach spaces.

Let $f \not\equiv 0$ be a holomorphic function on an open connected neighborhood of $\zeta \in \mathbf{B}_n$. Then a series $f(z) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(z-\zeta)$ converges uniformly on some neighborhood of ζ and represents f on this neighborhood. Here $\mathcal{P}_j$ is a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ, is called the zero multiplicity, or zero divisor of f at ζ, denoted by $\mathrm{div}_f(\zeta)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$; that is, a discrete set $\{\zeta_k\} \subset \mathbf{B}_n$ with $|\zeta_k| \to 1$ together with a sequence $\{m_k\}$ of positive integers. Associated to V, there is a unique closed ideal in $\mathbf{H}(\mathbf{B}_n)$,

$$J = J(V) := \{f \in \mathbf{H}(\mathbf{B}_n) : \mathrm{div}_f(\zeta_k) \geq m_k, \forall k\}.$$

Two holomorphic functions g, h in $H(\mathbf{B}_n)$ can be identified modulo J if and only

$$\frac{\partial^{|I|} g(\zeta_k)}{\partial z^I} = \frac{\partial^{|I|} h(\zeta_k)}{\partial z^I}, 0 \leq |I| < m_k, k \in \mathbf{N},$$

here and throughout the paper, we use I to denote a muti-index; that is, $I = (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$. The quotient space $\mathbf{H}(\mathbf{B}_n)/J$ can be identified to the space $\mathbf{H}(V)$ of all sequences $\{a_{k,I}\}_{k\in\mathbf{N}, 0\leq|I|<m_k}$ of complex numbers, which can be described as "analytic functions " on V. The map

$$\rho : \rho(f) = \{\frac{\partial^{|I|} f(\zeta_k)}{I!\partial z^I}\}_{k\in\mathbf{N}, 0\leq|I|<m_k}$$

is the natural restriction map from $\mathbf{H}(\mathbf{B}_n)$ into $\mathbf{H}(V)$.

Now we are going to define the space $l^{-\infty}(V)$ of "analytic functions" with growth conditions on a multilicity variety V. This will be the range of $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ under the restriction map ρ in interpolation.

Definition 2.2. Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$. Then

$$l^{-\infty}(V) := \{\{a_{k,I}\}_{k\in\mathbf{N}, 0\le|I|<m_k} : \sup_{k\in\mathbf{N}}(1-|\zeta_k|)^A \sum_{|I|=0}^{m_k-1} |a^*_{k,I}| < \infty$$

for some $A > 0\}$, where $a^*_{k,I} := (\lambda(1-|\zeta_k|))^{|I|} a_{k,I}$ is the "correction" of $a_{k,I}$ and $0 < \lambda < \frac{1}{n}$ is a constant, which is fixed throughout the paper.

Proposition 2.3. $\rho(\mathbf{A}^{-\infty}(\mathbf{B}_n)) \subset l^{-\infty}(V)$.

Proof. Let $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then there exist $A, B > 0$ such that $|f(z)| \le \frac{A}{(1-|z|)^B}$. Since $\lambda < \frac{1}{n}$, there exists a α such that $0 < \lambda n < \alpha < 1$ and so that $\frac{\lambda}{\alpha} < \frac{1}{n}$. Thus, there exists a $\epsilon > 0$ such that

$$\frac{\lambda}{\alpha} \le \frac{1}{n+\epsilon}. \tag{2.1}$$

Consider

$$g(z) = f(\zeta_k + \alpha(1-|\zeta_k|)z), z \in \mathbf{B}_n.$$

Then we see that

$$|g(z)| \le \frac{A}{[1-(|\zeta_k| + \alpha(1-|\zeta_k|))]^B} = \frac{A}{(1-\alpha)^B(1-|\zeta_k|)^B}. \tag{2.2}$$

Note that g is holomorphic in $\mathbf{B}_n$ and continuous up to the boundary S of $\mathbf{B}_n$. By the Cauchy formula in the uniut ball (see e.g. [R]) we have

$$g(z) = \int_S \frac{g(w)}{(1- < w, z >)^n} d\sigma(w),$$

where σ is the normalized rotation-invariant positive Borel measure on S and $< w, z >$ is the usual inner product. Thus for $I = (i_1, i_2, \cdots, i_n)$, we have that

$$\frac{\partial^{|I|} g(z)}{\partial^I z} = (-1)^{|I|} n(n+1)\cdots(n+|I|-1) \int_S \frac{w_1^{i_1}\cdots w_n^{i_n}}{(1- < z, w >)^{n+|I|}} d\sigma(w),$$

where $w = (w_1, \cdots, w_n)$, from which we obtain that

$$\begin{aligned} & |\frac{\partial^{|I|} g(0)}{\partial^I z}| \\ & \leq n(n+1)\cdots(n+|I|-1) \int_S |g(w)| d\sigma(w) \\ & \leq \frac{An(n+1)\cdots(n+|I|-1)}{(1-\alpha)^B (1-|\zeta_k|)^B} \end{aligned}$$

in view of (2.2) and the fact that $\int_S d\sigma(w) = 1$. But

$$(\alpha(1-|\zeta_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{\partial^I z}| = |\frac{\partial^{|I|} g(0)}{\partial^I z}|.$$

We obtain that

$$\begin{aligned} & \sum_{|I|=0}^{\infty} (\lambda(1-|\zeta_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}| \\ & \leq \frac{A}{(1-\alpha)^B (1-|\zeta_k|)^B} \sum_{|I|=0}^{\infty} (\frac{\lambda}{\alpha})^{|I|} \frac{n(n+1)\cdots(n+|I|-1)}{I!} \\ & \leq \frac{A}{(1-\alpha)^B (1-|\zeta_k|)^B} \sum_{|I|=0}^{\infty} (\frac{1}{n+\epsilon})^{|I|} \frac{n(n+1)\cdots(n+|I|-1)}{I!} \end{aligned}$$

in view of (2.1). We assert that the series

$$\sum_{|I|=0}^{\infty} (\frac{1}{n+\epsilon})^{|I|} \frac{n(n+1)\cdots(n+|I|-1)}{I!} < \infty.$$

To see this, consider the holomorphic function

$$h(z) = \frac{1}{\{1-(z_1+z_2+\cdots+z_n)\}^n}$$

in the polydisc $\mathbf{P} := \{z = (z_1, z_2, \cdots, z_n) : |z_1| < \frac{1}{n}, \cdots, |z_n| < \frac{1}{n}\}$. The function $h(z)$ can be expanded to the Taylor series in the polydisc $\mathbf{P}$ as follows:

$$h(z) = \sum_{|I|=0}^{\infty} \frac{(-1)^{|I|} n(n+1)\cdots(n+|I|-1)}{I!} z^I.$$

Noting that $z_0 := (-\frac{1}{n+\epsilon}, \cdots, -\frac{1}{n+\epsilon}) \in \mathbf{P}$, we obtain that

$$\sum_{|I|=0}^{\infty} (\frac{1}{n+\epsilon})^{|I|} \frac{n(n+1)\cdots(n+|I|-1)}{I!} = h(z_0) < \infty.$$

Thus, we have that

$$\sum_{|I|=0}^{\infty} (\lambda(1-|\zeta_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}| < \frac{A}{(1-\alpha)^B (1-|\zeta_k|)^B} h(z_0) < \infty$$

for some $A, B > 0$. Hence $\rho(f) \in l^{\infty}(V)$. The proof is complete. $\square$

Remark 2.4. In the above proposition we showed that for any $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$,

$$\sup_{k \in \mathbf{N}} (1-|\zeta_k|)^A \sum_{|I|=0}^{m_k-1} (\lambda(1-|\zeta_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}| < \infty$$

for some $A > 0$. This is a "precise" result for the unit ball $\mathbf{B}_n$ in the sense that the "correction factor" $(\lambda(1-|z_k|))^{|I|}$ in the above sum can not be dropped and $0 < \lambda < 1$ is best possible, and thus the above sequence space $l^{-\infty}(V)$ can not be replaced by the "natural-looking" space

$$A(V) := \{\{a_{k,I}\}_{k \in \mathbf{N}, 0 \le |I| < m_k} : \sup_{k \in \mathbf{N}} (1-|\zeta_k|)^A \sum_{|I|=0}^{m_k-1} |a_{k,I}| < \infty$$

for some $A > 0\}$. In fact, there exist a V and a f such that $\rho(f) \notin A(V)$ through $\rho(f) \in l^{-\infty}(V)$. This can be seen from the following

Proposition 2.5. *There exists a $f \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that for any $A > 0$,*

$$\sup_{k \in \mathbf{N}} (1-|\zeta_k|)^A \sum_{|I|=0}^{\infty} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}|$$

$$\ge \sup_{k \in \mathbf{N}} (1-|\zeta_k|)^A \sum_{|I|=0}^{\infty} (\frac{1}{n}(1-|\zeta_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}| = \infty$$

However, for any $0 < \lambda < \frac{1}{n}$ and $A \ge 1$,

$$\sup_{k \in \mathbf{N}} (1-|\zeta_k|)^A \sum_{|I|=0}^{\infty} (\lambda(1-|z_k|))^{|I|} |\frac{\partial^{|I|} f(\zeta_k)}{I! \partial^I z}| < \infty.$$

Proof. For the sake of convenience, we look at the case $n = 1$. Consider the function $f(z) = \frac{1}{1-z}$, which is holomorpphic in the unit disk. Let $V = \{\zeta_k\} = \{1 - \frac{1}{k}\}, k = 1, 2, \cdots$. Then $|\frac{f^{(j)}(\zeta_k)}{j!}| = k^{j+1}$. Thus for any $A > 0$,

$$\sup_{k\in \mathbf{N}}(1-|\zeta_k|)^A \sum_{j=0}^{\infty}(\frac{1}{n}(1-|\zeta_k|))^j|\frac{f^{(j)}(\zeta_k)}{j!}|$$

$$= \sup_{k\in \mathbf{N}}(\frac{1}{k})^A \sum_{j=0}^{\infty}(\frac{1}{k})^j k^{j+1} = \infty.$$

However, for each $0 < \lambda < 1$ and $A \geq 1$, we have that

$$\sup_{k\in \mathbf{N}}(1-|\zeta_k|)^A \sum_{j=0}^{\infty}(\lambda(1-|\zeta_k|)^j\frac{|f^{(j)}(\zeta_k)|}{j!}$$

$$= \sup_{k\in \mathbf{N}}(\frac{1}{k})^A \sum_{j=0}^{\infty}(\frac{\lambda}{k})^j k^{j+1}$$

$$= \sup_{k\in \mathbf{N}} k^{1-A} \sum_{j=0}^{\infty}\lambda^j < \infty. \quad \square$$

We have seen that $\rho(\mathbf{A}^{-\infty}(\mathbf{B}_n)) \subset l^{-\infty}(V)$, but in general, the space $l^{-\infty}(V)$ is too large. The interpolation problem with multiplicity stated in the introduction is to determine when ρ is surjective from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $l^{-\infty}(V)$. That is, under what conditions, is it true that for any multi-indexed sequence $\{a_{k,I}\} \in l^{-\infty}(V)$ there exists a holomorphic function in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ such that $f_{k,I} = a_{k,I}$ for any $k \in \mathbf{N}$ and $0 \leq |I| < m_k$; i.e., f has a described finite collection of Taylor coefficients. When $m_k = 1$ for all k, then $f_{k,I} = a_{k,I}$ simply means that $f(\zeta_k) = a_k$, where $\{a_k\}$ is a sequence satisfying that $\sup_{k\in\mathbf{N}}\{(1-|\zeta_k|)^A|a_k|\} < \infty$ for some constant $A > 0$.

Definition 2.6. A multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if the restriction map ρ is surjective from $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ to $l^{-\infty}(V)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety. We use $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \cdots, f_m)$, to denote that each F_j vanishes at ζ_k with multiplicity at least m_k; i.e., $\mathrm{div}_f(\zeta_k) \geq m_k$, $\forall k$. Given $\epsilon, C > 0$, we define $S(F; \epsilon, C)$ by (1.2), which can be thought as a "tube" of the variety V.

We shall prove the following theorem:

Theorem 2.7. *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{B}_n$ and $m \geq n$ a positive integer. Then V is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ if and only if there exist m functions $f_1, f_2, \cdots, f_m$ in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ and two constants $\epsilon, C > 0$ such that $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \cdots, f_m)$, and each connected component of $S(F; \epsilon, C) := \{z \in \mathbf{B}_n : |F(z)| < \epsilon(1-|z|)^C\}$ contains at most one point in V and the component containg z_k is of dimater at most $\lambda(1 - |z_k|)$.*

§**3. Some Lemmas.** In the following, we shall use A, B, C, ϵ to denote positive constants, the actual values of which may vary from one occurrence to the next. The number λ is the fixed constant given in Definition 2.2.

To prove the results, we need the following lemmas.

Lemma 3.1. *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then given $M > 0$ there exist two constants $l > 0$ and $\epsilon > 0$ such that*

$$A_{p,l}(V) \supset \{a = \{a_{k,I}\}_{k \in \mathbf{N}, |I| < m_k} : ||a|| < \epsilon\},$$

where

$$||a|| = \sup_{k \in \mathbf{N}} (1 - |\zeta_k|))^M \{ \sum_{|I|=0}^{m_k - 1} |a^*_{k,I}| \},$$

$$a^*_{k,I} = (\lambda(1 - |\zeta_k|))^I a_{k,I},$$

$$A_{p,l}(V) = \{a_f := \{f_{k,I}\}_{k \in \mathbf{N}, |I| < m_k} : f \in A_{p,l}(\mathbf{B}_n), ||a_f|| \leq 1\},$$

and

$$A_{p,l}(\mathbf{B}_n) = \{f \in \mathbf{A}^{-\infty}(\mathbf{B}_n) : (1 - |z|)^l |f(z)| \leq l, z \in \mathbf{B}_n\}.$$

Proof. Let

$$\mathcal{A} = \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| \le 1\}.$$

Then it is easy to check that $\mathcal{A}$ is complete under the metric induced by the norm $||a||$. Because V is an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, for any sequence $a = \{a_{k,I}\} \in \mathcal{A}$, there exists a $f \in A_{p,l}(\mathbf{B}_n)$ for some l such that $f_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $|I| < m_k$. That is, $a \in A_{p,l}(V)$. This shows that $\mathcal{A} = \cup_{l=1}^{\infty} A_{p,l}(V)$.

One can check that each $A_{p,l}(V)$ is a closed subset of $\mathcal{A}$. In fact, if $\{f_j\}$ is a sequence in $A_{p,l}(\mathbf{B}_n)$ such that $(f_j)_{k,I} \to a \in \mathcal{A}$ as $j \to \infty$, then by the definition of $\mathbf{A}^{-\infty}(\mathbf{B}_n)$, $\{f_j\}$ is uniformly bounded on each closed subset of $\mathbf{B}_n$. Using Montel's theorem (see e.g. [G]) we know that $\{f_j\}$ is a normal family in $\mathbf{B}_n$. By passing to a subsequence, we can assume that $f_j \to f$ normally, where f is the limit function. By the Weierstrass theorem and the uniqueness of the limit (see e.g. [G]), we deduce that $f \in A_{p,l}(\mathbf{B}_n)$ and $\{f_{k,I}\} = a$. It follows that $a \in A_{p,l}(V)$ and thus that $A_{p,l}(V)$ is closed. Now by the well-known Baire-category theorem, we know that for some l, $A_{p,l}(V)$ has a non-empty interior. Therefore, there exists a $\epsilon > 0$ such that $A_{p,l}(V) \supset \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$. □

Lemma 3.2. *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $A_p(\mathbf{B}_n)$ and $M > 0$. Then there exist two constants $l > 0$ and $\epsilon > 0$ such that the following two conclusions hold:*

(i) There exisits a sequence $\{f_k\}$ of holomorphic functions in $\mathbf{B}_n$ such that

$$(1-|z|)^l |f_k(z)| \le l, \quad z \in \mathbf{B}_n \quad \textit{and} \quad k \in \mathbf{N} \tag{3.1}$$

and $(f_k)_{j,I} = 0$ for all j and $|I| \le m_j - 1$ except that

$$\frac{\partial^{m_k-1} f_k(\zeta_k)}{(m_k-1)\partial z_1^{m_k-1}} = \frac{\epsilon}{(\lambda(1-|\zeta_k|))^{m_k-1}}; \tag{3.2}$$

(ii) There exists a sequence $\{g_k\}$ of holomorphic functions in $\mathbf{B}_n$ such that each g_k satisfies (3.1) and $(g_k)_{i,I} = 0, \quad \forall i, |I| < m_i - 1$ except that

$$g_k(\zeta_k) = \epsilon. \tag{3.3}$$

Proof. It follows from Lemma 3.1 that there exist two constants $l > 0$ and $\epsilon > 0$ such that the space $A_{p,l}(V)$ contains the space $\mathcal{S} := \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$ (see Lemma 3.1 for the notations). For each fixed k, consider the sequence $a_k = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k}$ satisfying that $a_{j,I} = 0$ for all j and $0 \leq |I| < m_j$ except that

$$a_{k,I_k} = \frac{\epsilon}{(\lambda(1-|\zeta_k|))^{m_k-1}},$$

where $I_k = (m_k - 1, 0, \cdots, 0) \in (\mathbf{Z}^+)^n$. Then it is clear that $a_k \in \mathcal{S}$. Hence there exist holomorphic functions f_k in $\mathbf{B}_n$ such that (3.1) holds and that $(f_k)_{j,I} = a_{j,I}$ for all j and $0 \leq |I| < m_j$. Thus (3.2) holds. The conclusion (ii) follows from the smae argument. □

Lemma 3.3 (Schwarz, [G]). *If f is holomorphic in an open neighborhood of a closed ball $\bar{B}(\zeta, r)$ in $\mathbf{C}^n$ centered at ζ and with radius r, $|f(z)| \leq M$ for $z \in B(\zeta, r)$, and $\frac{\partial^{|I|} f}{\partial z^I}(\zeta) = 0$ whenever $|I| < m$ for some $m \in \mathbf{N}$, then $|f(z)| \leq M r^{-m} |z - \zeta|^m$ for $z \in \bar{B}(\zeta, r)$.*

Lemma 3.4. *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. Then*

(i) $\lambda^{m_k} \geq \epsilon(1 - |\zeta_k|)^C$ for some $\epsilon, C > 0$, where λ is the fixed constant in Definition 2.2;

(ii) $\sum_{k=1}^{\infty} (1 - |\zeta_k|)^M < \infty$ for large $M > 0$.

Proof. By Lemma 3.2 (i) there exists a sequence of funcctions $\{f_k\}$ satisfying

(3.1) and (3.2). By the Cauchy Theorem,

$$\frac{1}{(m_k-1)!}\frac{\partial^{m_k-1}f_k(\zeta_k)}{\partial z_1^{m_k-1}} = (\frac{1}{2\pi i})^n \int \frac{f_k(z)dz_1\wedge dz_2\cdots\wedge dz_n}{(z_1-\zeta_{1,k})^{m_k}(z_2-\zeta_{2,k})\cdots(z_n-\zeta_{n,k})},$$

where $\zeta_k = (\zeta_{1,k}, \zeta_{2,k}, \cdots, \zeta_{n,k})$, and the integral is taken over the boundary of the polydisc

$$P := \{z : |z_1-\zeta_{1,k}| < \lambda^{\frac{1}{2}}(1-|\zeta_k|), \cdots, |z_n-\zeta_{n,k}| < \lambda^{\frac{1}{2}}(1-|\zeta_k|)^{\frac{1}{2}}\}$$

$$\subset \{z : |z-\zeta_k| < (n\lambda)^{\frac{1}{2}}(1-|\zeta_k|)\}.$$

Note that for $z\in P$,

$$|f_k(z)| \le \frac{l}{(1-|z|)^l} \le \frac{A}{(1-|\zeta_k|)^B}$$

for some $A, B > 0$ since

$$1-|z| \ge 1-(|z-\zeta_k|+|\zeta_k|) \ge (1-(n\lambda)^{\frac{1}{2}})(1-|\zeta_k|).$$

We have, in view of (3.2), that

$$\frac{\epsilon}{(\lambda(1-|\zeta_k|))^{m_k-1}} \le \frac{A}{(1-|\zeta_k|)^B}\frac{1}{(\lambda^{\frac{1}{2}}(1-|\zeta_k|))^{m_k-1}}$$

and so that $\lambda^{m_k} \ge \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$. That is, (i) holds.

For each k, there exists a j such that $d_k = |\zeta_k - \zeta_j|$, where $d_k := \inf_{j\ne k}\{|\zeta_j - \zeta_k|\}$. Let g_k be the functions in Lemma 3.2 satisfying (3.1) and (3.3). Set $h_k(z) = g_k(z) - g_k(\zeta_k)$. Then $h_k(\zeta_k) = 0$. When $z \in U := \{z : |z-\zeta_k| < \lambda(1-|\zeta_k|)\}$ we have that

$$h_k(z) \le \frac{l}{(1-|z|)^l} + \epsilon \le \frac{A}{(1-|\zeta_k|)^B}$$

and thus that, using Lemma 3.3,

$$|h_k(z)| \le \frac{A|z-\zeta_k|}{(1-|\zeta_k|)^B}.$$

Hence, if $\zeta_j \in U$ we will have that

$$\epsilon = |g_k(\zeta_k)| = |h_k(\zeta_j)| \leq \frac{A|\zeta_k - \zeta_j|}{(1-|\zeta_k|)^B}$$

and so that $|\zeta_j - \zeta_k| \geq \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$. If $\zeta_j \notin U$, then $|\zeta_j - \zeta_k| \geq \lambda(1-|\zeta_k|)$. Therefore, in any case we always have that $d_k \geq \epsilon(1-|\zeta_k|)^C$ for some $0 < \epsilon < 1$ and $C > 0$. Let B_k be the ball centered at ζ_k with radius d_k. Then $B_k \cap B_j = \emptyset$ for $i \neq j$ and the volume $|B_k|$ of B_k satisfies that $|B_k| \geq \epsilon(1-|\zeta_k|)^C$ for some $\epsilon, C > 0$. Thus, we deduce that

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|\zeta_k|)^M &= \sum_{k=1}^{\infty}\frac{1}{|B_k|}\int_{B_k}(1-|\zeta_k|)^M dm \\ &\leq A\sum_{k=1}^{\infty}\int_{B_k}(1-|\zeta_k|)^{M-C} dm \\ &\leq A\sum_{k=1}^{\infty}\int_{B_k}(1-|z|)^{M-C} dm \\ &\leq A\int_{\mathbf{B}_n}(1-|z|)^{M-C} dm < \infty\end{aligned}$$

for large M. This completes the proof. □

§4. **Proof of Theorem 2.7.** We first prove the necessity. Given a $M > 0$, by Lemma 3.1, there exists a positive integer l and a ϵ_0 such that

$$A_{p,l}(V) \supset \{\{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : \sup_{k\in\mathbf{N}}\{(1-|\zeta_k|))^M \sum_{|I|=0}^{m_k-1}|a^*_{k,I}|\} \leq \epsilon_0\}.$$

Here we use the same notations as in Lemma 3.1. Thus, for each $1 \leq j \leq n$ we can obtain sequences of holomorphic functions $\{g_{j,k}\}$ with $g_{j,k} \in A_{p,l}(\mathbf{B}_n)$ for any $k \in \mathbf{N}$ and $1 \leq j \leq n$ such that $(g_{j,k})_{i,I} = 0, \quad \forall i, |I| < m_i$ except that

$$\frac{\partial^{l_k} g_{j,k}}{\partial z_j^{l_k}}(\zeta_k) = \frac{\epsilon_0}{(\lambda(1-|\zeta_k|))^{l_k}(1-|\zeta_k|)^M} \tag{4.1}$$

where $l_k = \frac{m_k}{2}$ if m_k is even and $l_k = \frac{m_k-1}{2}$ if m_k is odd. We define, for $1 \le j \le n$, the following functions

$$f_j(z) = \sum_{k=1}^{\infty} h_{j,k}(1 - |\zeta_k|)^{2M}, \tag{4.2}$$

where $h_{j,k} = g_{j,k}^2(z)$ if m_k is even and $h_{j,k} = \frac{(z_j - \zeta_{j,k})g_{j,k}^2(z)}{\lambda(1-|\zeta_k|)}$ if m_k is odd, $z = (z_1, \cdots, z_n)$ and $\zeta_k = (\zeta_{1,k}, \cdots, \zeta_{n,k})$. It is clear that $\mathrm{div}_{f_j}(\zeta) \ge m_k$ and so that $V \subset F^{-1}(0)$, where $F = (f_1, \cdots, f_n)$. We claim that $f_j \in A_p(\mathbf{B}_n)$ for each $1 \le j \le n$. In fact, since $g_{j,k} \in A_{p,l}(\mathbf{B}_n)$ for any $k \in \mathbf{N}$, we have that for $z \in \mathbf{B}_n$, $|g_{j,k}(z)| \le \frac{l}{(1-|z|)^l}$. Therefore, we deduce that

$$|g_{j,k}^2(z)|(1 - |\zeta_k|)^{2M} \le \frac{l^2(1 - |\zeta_k|)^{2M}}{(1 - |z|)^{2l}}.$$

By Lemma 3.4 (ii), taking a M sufficiently large, we see that the series (3.2) is uniformly convergent in closed sets of $\mathbf{B}_n$, and moreover $|f_j(z)| \le \frac{A}{(1-|z|)^{2l}}, z \in \mathbf{B}_n$ for some constants $A > 0$; that is , $f_j \in A_p(\mathbf{B}_n)$.

Next we show that there are positive constants ϵ, C such that the "tube" $S(F; \epsilon, C)$ satisfies the conclusion of the theorem. To this end, let $k > 0$ and let $u = (u_1, \cdots, u_n)$ be a unit vector in $\mathbf{C}^n$. Then there exists a j $(1 \le j \le n)$ such that $u_j \ge \frac{1}{\sqrt{n}}$. For this fixed j, consider the Taylor expansion of $f_j(z)$ at ζ_k. One can verify that, in view of (4.1) and (4.2),

$$f_j(z) = \epsilon_0^2(\lambda(1 - |\zeta_k|))^{-m_k}(z_j - \zeta_{j,k})^{m_k} + \sum_{|I| \ge m_k + n_k}^{\infty} C_I(z - \zeta_k)^I,$$

where $n_k = \frac{m_k}{2}$ if m_k is even and $n_k = \frac{m_k+1}{2}$ if m_k is odd, C_I's are complex numbers. From the above expansion, we deduce that for $w \in \mathbf{B}_1 \subset \mathbf{C}$,

$$F_j(w) := f_j(\zeta_k + u\sqrt{\lambda}(1 - |\zeta_k|)w) = \epsilon_1 w^{m_k} + \sum_{j \le m_k + n_k} b_j w^j, \tag{4.3}$$

where $\epsilon_1 = \epsilon_0^2 u_j^{m_k} \sqrt{\lambda}^{-m_k}$ and b_j's are complex numbers. Noting that $\lambda < \frac{1}{n}$, we obtain that

$$\epsilon_1 \geq \epsilon_0^2 (\frac{1}{\sqrt{n}})^{m_k} (\frac{1}{\sqrt{n}})^{-m_k} = \epsilon_0^2. \tag{4.4}$$

Let

$$d_u = \min\{1, \text{dist } (0, F_j^{-1}(0) \setminus \{0\})\}$$

and set $G_j(w) = \frac{F_j(w)}{w^{m_k}}$. Then $|G_j(w)| \leq \frac{A}{(1-|\zeta_k|)^B}$ for some constants $A, B > 0$ on $|w| = 1$ and thus in $|w| \leq 1$ by the maximum modulus theorem. Also let $H_j(w) = G_j(w) - G_j(0)$. Then by (4.3), we see that $H_j(w)$ has a zero at $w = 0$ of order at least n_k. Note that $|H_j(w)| \leq \frac{A}{(1-|\zeta_k|)^B}$ for some constants $A, B > 0$ on $|w| \leq 1$. We have , by Lemma 3.3, that

$$|H_j(w)| \leq \frac{A}{(1 - |\zeta_k|)^B} |w|^{n_k}$$

on $|w| \leq 1$. Thus, if $a \neq 0$ is a zero of $F_j(w)$ in $|w| < 1$, then $G_j(a) = 0$ and thus that

$$|H_j(a)| = |G_j(0)| = \epsilon_1 \geq \epsilon_0^2$$

by (4.3) and (4.4), from which it follows that

$$|a|^{n_k} \geq |H_j(a)| A^{-1} (1 - |\zeta_k|)^B$$

and thus that $d_u^{n_k} > \epsilon(1 - |\zeta_k|)^C$ for some constants $\epsilon, C > 0$, which implies that

$$d_u^{m_k} > \epsilon(1 - |\zeta_k|)^C.$$

Therefore,

$$d_u > \epsilon^{\frac{1}{m_k}} (1 - |\zeta_k|)^{\frac{C}{m_k}} := \frac{1}{\lambda} d_k, \tag{4.5}$$

where $0 < \epsilon < 1$ and C are two constants. Note that $G_j(w)$ has no zero in $|w| \leq \frac{1}{\lambda} d_k$ by the construction of d_u. Recall the following result from the

Carathéodory theorem (see e.g. [L]): If h is holomorphic and has no zero in $|w| \leq R$ with $h(0) = 1$, then $\log|h(w)| \geq -\frac{2r}{R-r}\log\max_{|w|=R}\{|h(w)|\}$ for $|w| \leq r < R$. Applying it to $G_j(w)$ in $|w| \leq \frac{1}{\lambda}d_k$ we deduce that for $|w| \leq d_k$

$$\log|\frac{G_j(w)}{G_j(0)}| \geq -\frac{2\lambda}{1-\lambda}\log(\max_{|w|=d_k}\{|\frac{G_j(w)}{G_j(0)}|\}),$$

which implies that $|G_j(w)| \geq \epsilon(1-|\zeta_k|)^C$ for some constants $\epsilon, C > 0$. In particular, for $|w| = d_k$,

$$\begin{aligned}|F_j(w)| &= |w^{m_k}G_j(w)| \\ &\geq \{\lambda\epsilon^{\frac{1}{m_k}}(1-|\zeta_k|)^{\frac{C}{m_k}}\}^{m_k}\epsilon(1-|\zeta_k|)^C \geq \epsilon(1-|\zeta_k|)^C\end{aligned}$$

for some $\epsilon, C >$) by virtue of (4.5) and Lemma 3.4(i).

So far we have proved that for a given unit vector $u \in \mathbf{C}^n$, there exists a j $(1 \leq j \leq n)$ such that $|f_j(\zeta_k + u\sqrt{\lambda}(1-|\zeta_k|)w)| \geq \epsilon(1-|\zeta_k|)^C$ on $|w| = d_k$, where the constants ϵ, C are independent of u and k. Therefore, for $z \in \mathbf{B}_n$ with $|z-\zeta_k| = \sqrt{\lambda}(1-|\zeta_k|)d_k$, we always have that

$$|F(z)| = (\sum_{j=1}^{n}|f_j(z)|^2)^{\frac{1}{2}} \geq \epsilon(1-|\zeta_k|)^C$$

for some $\epsilon, C > 0$. Now consider the neighborhood $U_k := \{z \in \mathbf{C}^n : |z-\zeta_k| \leq \sqrt{\lambda}(1-|\zeta_k|)d_k\}$ of ζ_k. By the above result, we know that $|F(z)| \geq \epsilon(1-|\zeta_k|)^C$ on ∂U_k. Recall that $S(F;\epsilon,C) = \{z \in \mathbf{C}^n : |F(z)| < \epsilon(1-|\zeta_k|)^C\}$. Thus the connected component V_k of $S(F;\epsilon,C)$ containing ζ_k is clearly contained in U_k. By the construction of d_k, we see that U_k, and thus V_k, has diameter less than $\lambda(1-|\zeta_k|)$ and does not contain other points of V. (If $m > n$, we can easily add $m-n$ entire functions $f_{n+1}, \cdots, f_m$ so that $f_1, f_2, \cdots, f_m$ satisfy the conclusion of the theorem.) This completes the proof of the necessity.

To prove the sufficiency, let V_k be the connected component of $S(F;\epsilon,C)$ containing ζ_k. Suppose that $\{a_{k,I}\} \subset l^{-\infty}(V)$ be a given multi-indexed sequence with

$$\sum_{|I|=0}^{m_k-1} (\lambda(1-|\zeta_k|))^I |a_{k,I}| \leq \frac{A}{(1-|\zeta_k|)^B}$$

for some constants $A, B > 0$. We define an analytic function $\gamma : S(F;\epsilon, C) \to \mathbf{C}$ by

$$\gamma(z) = \begin{cases} \sum_{|I|=0}^{m_k-1} a_{k,I}(z-\zeta_k)^I, & \text{if } z \in V_k; \\ 0, & \text{if } z \in S(F;\epsilon,C) \setminus \cup_{k\in\mathbf{N}} V_k. \end{cases}$$

Then it is clear that $\gamma_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \leq |I| \leq m_k - 1$. Since $|z-\zeta_k| \leq \lambda(1-|\zeta_k|)$ on V_k by the assumption, we see that, for $z \in S(F;\epsilon, C)$,

$$|\gamma(z)| \leq \sum_{|I|=0}^{m_k-1} (\lambda(1-|\zeta_k|))^I |a_{k,I}| \leq \frac{A}{(1-|\zeta_k|)^B}.$$

Note that for $z \in S(F, \epsilon, C)$,

$$1 - |\zeta_k| \geq 1 - (|\zeta_k - z| + |z|) \geq (1-|z|) - \lambda(1-|\zeta_k|)$$

or $1 - |\zeta_k| \geq \frac{1-\lambda}{1-|z|}$ by the assumption. We deduce that

$$|\gamma(z)| \leq \frac{A}{(1-|z|)^B} \tag{4.6}$$

for some $A, B > 0$ for $z \in S(F, \epsilon, C)$ and thus for $z \in \mathbf{B}_n$. We will extend γ to a holomorphic function in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$ by the L^2-estimate for $\bar{\partial}$ equations (cf. [BT], [H] and [KT]). Since $\frac{\partial f_j}{\partial z_i} \in \mathbf{A}^{-\infty}(\mathbf{B}_n)$,

$$\sum_{i=1}^{n} |\frac{\partial f_j}{\partial z}(z)| \leq \frac{A}{(1-|z|)^B}$$

for $z \in \mathbf{B}_n$ and some $A, B > 0$. Take a small $\epsilon_1 < \epsilon$ and a large $C_1 > C$. We assert that the distance $d(z)$ of a point $z \in S(F;\epsilon_1, C_1)$ to the complement of $S(F;\epsilon, C)$ satisfies that

$$d(z) \geq \epsilon_2(1-|z|)^{C_2} \tag{4.7}$$

for some small $\epsilon_2 < \epsilon_1$ and large $C_2 > C_1$. Otherwise there would be a w on the boundary of $(F; \epsilon, C)$ such that $|w - z| \le \epsilon_2(1 - |z|)^{C_2}$. Then

$$|f(w) - f(z)| = |\int_0^1 \frac{df}{dt}(z + (w - z)t)dt|$$
$$\le |w - z| \int_0^1 |\sum_{i=1}^{n} \frac{\partial f_i}{\partial z_i}(z + (w - z)t)|$$
$$\le \epsilon_2(1 - |z|)^{C_2} \frac{A}{(1 - |z|)^B}.$$

Then

$$|f(w)| \le |f(z)| + |f(z) - f(w)|$$
$$\le \epsilon_1(1 - |z|)^{C_1} + \epsilon_2(1 - |z|)^{C_2} \frac{A}{(1 - |z|)^B}$$
$$\le \epsilon(1 - |z|)^C$$

if ϵ_1, ϵ_2 are taken sufficiently small and C_1, C_2 sufficiently large. It is contradict to the choice of the point of w. Now we can choose a cut-off function $\chi \in C^\infty$ (see [BG1, p18]) such that $0 \le \chi \le 1$,

$$|\bar{\partial}\chi| \le A(d(z))^B \le \frac{A}{(1 - |z|)^B}$$

in view of 4.7, $\chi = 1$ on $S(F; \epsilon_1, C_1)$ and $\chi = 0$ on a neighborhood of the complement of $S(F; \epsilon, C)$. Then $\phi := \gamma\bar{\partial}\chi$ is a $\bar{\partial}$ closed form. By virtue of (4.6) and the fact that $|F(z)| \ge \epsilon(1 - |z|)^C$ for $z \in \mathrm{supp}(\phi)$, for each $\alpha > 0$ there exist a $\beta > 0$ such that

$$\int_{\mathbf{B}_n} \frac{|\phi(z)|^2}{|F(z)|^\alpha}(1 - |z|)^\beta dm < \infty.$$

By theorem 2.6 in [KT] there exist $\bar{\partial}$ closed $(0,1)$-forms $\phi_1, \phi_2, \cdots, \phi_m$ and some $q > 0$ such that $\phi = \phi_1 f_1 + \cdots + \phi_m f_m$ and

$$\int_{\mathbf{B}_n} |\phi_j(z)|^2(1 - |z|)^q dm < \infty.$$

Thus by Hömander's theorem [H], there exist solutions ψ_j to the $\bar{\partial}$ equations $\bar{\partial}\psi_j = \phi_j$ satisfying the L^2-estimate:

$$\int_{\mathbf{B}_n} |\psi(z)|^2(1 - |z|)^q dm < \infty.$$

Define $f = \gamma\chi - \sum_{j=1}^{m} \psi_j f_j$. Then

$$\bar{\partial} f = \gamma\bar{\chi} - \sum_{j=1}^{m} f_j \bar{\psi}_j = \phi - \sum_{j=1}^{m} f_j \phi_j = 0$$

and furthermore

$$\int_{\mathbf{B}_n} |f(z)|^2 (1-|z|)^A dm < \infty,$$

for some $A > 0$, which implies that f is in one of the weighted Bergman space and thus in $\mathbf{A}^{-\infty}(\mathbf{B}_n)$. By checking its Taylor expansion about ζ_k, we easily see that $f_{k,I} = \gamma_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \leq |I| \leq m_k - 1$. This shows that V is an interpolating variety for $A_p(\mathbf{C}^n)$. □

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