DISCRETENESS AND OPENNESS FOR MAPPINGS OF FINITE DISTORTION IN THE CRITICAL CASE $p = n - 1$

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ABSTRACT.

Let $F \in W^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ be a mapping with non-negative Jacobian $J_F(x) = \det DF(x) \geq 0$ a.e. in a domain $\Omega \in \mathbb{R}^n$. The dilatation of the mapping F is defined, almost everywhere in Ω, by the formula

$$K(x) = \frac{|DF(x)|^n}{J_F(x)}.$$

If $K(x)$ is bounded a.e., the mapping is said to be quasiregular. Quasiregular mappings are a generalization to higher dimensions of holomorphic mappings. The theory of higher dimensional quasiregular mappings began with Reŝhetnyak's theorem, stating that non constant quasiregular mappings are continuous, discrete and open.

In some problems appearing in the theory of non-linear elasticity, the boundedness condition on $K(x)$ is too restrictive. Tipically we only know that F has finite dilatation, that is, $K(x)$ is finite a.e. and $K(x)^p$ is integrable for some value p. In two dimensions, Iwaniec and Šverak [IS] have shown that $K(x) \in L^1_{loc}$ is sufficient to guarantee the conclusion of Reŝhetnyak's theorem.

For $n \geq 3$, Heinonen and Koskela [HK], showed that if the mapping is quasi-light and $K(x) \in L^p_{loc}$ for $p > n-1$, then the mapping $F(x)$ is continuous, discrete and open. Manfredi and Villamor [MV] proved a similar result without assuming that the mapping $f(x)$ was quasi-light. The result is known to be false, see [Ball], when $p < n-1$.

In this paper we attempt to improve in those results. In particular, we will deal with the case $p = n-1$ for $n \geq 3$, and will assume that our mapping $F(x)$ is quasi-light, that is, the inverse image of any point is compact in Ω.

Our approach will be different from the ones used in [MV] and [HK]. It is more geometrical in nature and uses the method of extremal length.

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§1. Introduction.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $F\colon \Omega \to \mathbb{R}^n$ be a mapping in the Sobolev space $W^{1,n}_{loc}(\Omega; \mathbb{R}^n)$ of mapping in $L^n_{loc}(\Omega; \mathbb{R}^n)$ whose distributional derivatives belong to $L^n_{loc}(\Omega; \mathbb{R}^n)$. The differential of F at a point x is denoted by $DF(x)$, its norm is defined by

$$\|DF(x)\| = \sup\{|DF(x)\,h| \colon h \in \mathbb{R}^n, \ \ \|h\| = 1\},$$

and its Jacobian determinant is defined as $J_F(x) = \det\, DF(x)$. We assume in the rest of this paper that F is orientation preserving, meaning that $J_F(x) \geq 0$ for a.e. $x \in \Omega$. The dilatation of F at a point x is defined by the ratio

$$K(x) = \frac{\|DF(x)\|^n}{J_F(x)}.$$

If $K(x) \in L^\infty(\Omega; \mathbb{R}^n)$, then F is said to be a quasiregular mapping. We will say that F is a mapping with finite dilatation if

$$1 \leq K(x) < \infty$$

for almost every $x \in \Omega$ that is, except possibly for a set of measure zero in Ω. We will follow the convention that for a.e. $x \in \Omega$ we have that if $J_F(x) = 0$, then $DF(x) = 0$.

A basic result in the theory of quasiregular mappings, states that they are discrete and open or constant. Vodopyanov and Goldstein [VG], proved that mappings of finite dilatation are continuous and have monotone components in the sende of Dirichlet. An example of J. Ball [Ball] shows that there are mappings satisfying that $K(x) \in L^p_{loc}$ for every $p < n - 1$ that fail to be discrete.

A theorem of Iwaniec and Šverak [IS], shows that in the plane, a mapping with integrable dilatation, $K(x) \in L^1_{loc}$, can be expressed as the composition of an analytic mapping with an homeomorphism. It follows from that, that the mapping is discrete and open. The proof in [IS] is based on the solution of the linear two dimensional Beltrami equation and does not generalize in an obvious way to higher dimensions. Iwaniec and Šverak show that for those mapping a Stoiliv's decomposition holds. They conjectured in their paper that for higher dimensions, if $K \in L^{n-1}_{loc}$ then F is discrete and open.

In the higher dimensional case, $n \geq 3$, Villamor and Manfredi [MV], proved that if $K \in L^p_{loc}$ for any $p > n - 1$ then F is discrete and open.

Our main goal in this paper is to prove the following result.

Theorem 1. *Let $F \in W^{1,n}_{loc}(\Omega; \mathbb{R}^n)$ be a quasi-light, nonconstant mapping whose dilatation $K(x)$ is in $L^{n-1}_{loc}(\Omega)$. Then the mapping F is discrete and open.*

The rest of the paper is constituted as follows. In section §2 we will go through some rpeliminaries that will be needed in the rest of the paper. Section §3 we will talk about some necessary conditions that enable us to use the change of variable formula for integrals. In Section §4 we will prove the boundedness of the counting multiplicity function $N(F, y; \Omega)$ for the mapping F and some other asymptotic conditions on it which will still ensure the conclusion of our main results. In Section §5 we will give a necessary condition for the general case (i.e. not requiring that the mapping F is quasi-light), which will allow us to prove the openess and discreteness of the mapping F. Section § will be devoted to show that the weight $w(y) = (\ln \ln \frac{1}{|y|})^n$ is an A_n weight in the sense of Muckhenhoupt. In section §7 we will prove that certain weighted variational $(n-1)$ capacity is zero for $F^{-1}(a)$, and finally in Section §8 we will show that from that it follows that the mapping F is discrete and open, which will end the proof of our main result.

§2. Preliminaries

2.1 Hausdörff Measures.

Let s be a positive number and $0 < \delta \leq \infty$. Let $E \subset \mathbb{R}^n$ we define

$$\Lambda_s^\delta(E) = \inf \sum_i r_i^s,$$

where the infimum is taken over all coverings of E by balls B_i with radius r_i not exceeding δ. Λ_s^δ is an outer measurewhich in general fails to be additive on families of disjoint compact sets. Therefore, we define the s-Hausdörff measure of E as

$$\Lambda_s(E) = \sup_{\delta>0} \Lambda_s^\delta(E) = \lim_{\delta \to 0+} \Lambda_s^\delta(E).$$

The measure Λ_s is a Borel regular measure. That is, it is an additive measure on the Borel sets of $\mathbb{R}^n$ and for each Lebesgue measurable set $E \subset \mathbb{R}^n$ there is a Borel set G such that $E \subset G$ and $\Lambda_s(G) = \Lambda_s(E)$.

For any set E, it is clear that $\Lambda_s(E)$ is a non-increasing function of s. Furthermore, if $s < t$, then

$$\Lambda_s^\delta(E) \geq \delta^{s-t}\, \Lambda_t^\delta(E),$$

which implies that if $\Lambda_t(E)$ is positive, then $\Lambda_s(E)$ is equal to infinity. Thus, there is a unique value, called the Hausdörff dimension of the set E ($\dim_H(E)$), such that $\Lambda_s(E) = \infty$ if $0 \leq s < \dim_H(E)$ and $\Lambda_s(E) = 0$ if $\dim_H(E) < s < \infty$. For our purposes we require however a slightly more general concept than the s-Hausdörff measure. Namely, let h be a real valued increasing function on the interval $[0, 1)$ with $\lim_{t\to 0} h(t) = 0$. We define the h-Hausdörff measure of E by

$$\Lambda_h(E) = \sup_{\delta>0}\ inf \sum_i h(r_i),$$

where the infimum is again taken over all the coverings of the set E by balls B_i with radius r_i not exceeding δ. The measures Λ_h are still Borel regular measures in $\mathbb{R}^n$, see [HKM] for more references. The choice $h(t) = t^s$ gives the s-Hausdörff measures Λ_s defined above.

2.2 Modulus.

In the following, by a curve we mean a non-point locally rectifiable curve in $\mathbb{R}^n$. Let Γ be a family of curves. We shall say that a Borel non-negative measurable function ρ is Γ-admissible if $\int_\gamma \rho\, ds \geq 1$ for evevry $\gamma \in \Gamma$, where s is the arc-length parameter of the curve γ.

Let w be a non-negative measurable function in $\mathbb{R}^n$. We define the weighted p-module of Γ by

$$M_P^w(\Gamma) = \inf\{\int_{\mathbb{R}^n} \rho^p\, w\, dx\colon \rho \text{ is } \Gamma - \text{ admissible}\}.$$

We say that a non-negative measurable function w is a weight satisfying the Muckenhoupt A_p condition in Ω if

$$\sup_{\mathbb{B}\subset\Omega}\{\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} w\, dx\}\,\{\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} w^{\frac{1}{(1-p)}}\, dx\}^{p-1} < \infty,$$

where $\mathbb{B}$ is a ball and $|\mathbb{B}|$ stands for its volume. We denote it by $w \in A_p(\Omega)$.

Let X be a bounded set in $\mathbb{R}^n$ containing the origin. We denote by $\Lambda(0)$ the family of rectifiable curves in X ending at 0. Then we have the following lemma.

Lemma 2.1. *Let $w \in A_p(\mathbb{R}^n)$. Then the modulus $M_p^w(\Lambda(0)) = 0$ if and only if*

$$\int_{|x|<1} |x|^{(1-n)p'}\, w(x)^{\frac{1}{(1-p)}}\, dx = \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

For a proof of this Lemma, see [HH].

2.2 Capacities. A good reference for all the results in this subsection is [HKM]. We pass to define the weighted variational p-capacity of a compact set $K \subset \Omega$.

Definition. *Let K be a compact subset of Ω. Define*

$$W(K,\Omega) = \{\mu \in C_0^\infty(\Omega)\colon \mu \geq 1\, in\, K\}.$$

We define the weighted variational p-capacity of K as follows

$$(p,w) - cap\,(K,\Omega) = \inf_{\mu\in W(K,\Omega)}\{\int_\Omega |\nabla\mu|^p\, w\, dx\}.$$

For any open set $U \subset \Omega$ we define

$$(p,w) - \text{cap}\,(U,\Omega) = \sup\{(p,w) - \text{cap}\,(K,\Omega) \colon K \subset U \text{ compact}\}$$

and finally, for an arbitrary set $E \subset \Omega$ we define

$$(p,w) - \text{cap}\,(K,\Omega) = \inf\{(p,w) - \text{cap}\,(K,\Omega) \colon E \subset U \subset \Omega,\ U \text{ open}\}.$$

The following two properties are immediate consequences of the definition.

(i) If $E_1 \subset E_2$, then we have that $(p,w) - \text{cap}\,(E_1,\Omega) \leq (p,w) - \text{cap}\,(E_2,\Omega)$, and

(ii) If $\Omega_1 \subset \Omega_2$ are open and $E \subset \Omega_1$, then we have that $(p,w) - \text{cap}\,(E,\Omega_2) \leq (p,w) - \text{cap}\,(E,\Omega_1)$.

A set $E \subset \mathbb{R}^n$ is said to be of variational (p,w)-capacity zero if $(p,w) - \text{cap}\,(E \cap \Omega, \Omega) = 0$ for all open sets $\Omega \subset \mathbb{R}^n$. In this case we will write that $(p,w) - \text{cap}\,(E) = 0$.

We will later need the following lemma, for reference see [HKM, p. 37].

Lemma 2.2. *There is a positive constant C independent of p, w, x_0 and τ, such that*

$$\frac{1}{C}\left(\int_{\mathbb{B}^n(x_0,r)} w(x)\,dx\right)\tau^{-p} \leq (p,w) - cap(\mathbb{B}(x_0,\tau),\mathbb{B}(x_0,2\,\tau)), \tag{(2.1)}$$

for some positive weights, see [HKM].

It is important to clarify that we could have taken the closure of the ball $\bar{\mathbb{B}}(x_0,\tau)$ instead, since although the (p,w)-cap$(G;\Omega)$ differs from (p,w)-cap$(\bar{G};\Omega)$ when G is an open subset of Ω, when $\mathbb{B}$ is an open ball such that $\bar{\mathbb{B}} \subset \Omega$ we have that (p,w)-cap$(\mathbb{B};\Omega) = (p,w)$-cap$(\bar{\mathbb{B}};\Omega)$.

The inequality (2.1) does not hold for all weights w. We will show that for a particular weight w, we can follow the steps of Lemma 2.2 in [HKM] and show that for this weight the inequality in Lemma 2.2 holds. Let us define our weight w as follows,

$$w(x) = \left(\ln\ln\frac{1}{|F(x)|}\right)^{n-1},$$

where we have that $|F(x)| < 1$ for all x.

Proof. To prove the inequality on (2.1) and according to [HKM], we need to show that the measure μ defined as $d\mu(x) = w(x)\,dx$, satisfies the following Poincare type inequality

$$\int_{\mathbb{B}^n(x_0,2r)} \eta(x)^{n-1}\,d\mu(x) \leq C\,r^{n-1}\int_{\mathbb{B}^n(x_0,2r)} |\nabla\eta(x)|^{n-1}\,d\mu(x),$$

for each $\eta(x) \in C_0^\infty(\mathbb{B}^n(x_0, 2r))$. C is a constant independent of x_0, r, η and w. In order to prove this, we will use a result proved in [MV]. Namely,

$$V(x) = \eta(x) \left(\ln \ln \frac{1}{|F(x)|} \right)$$

is a function in the Sobolev space $W_0^{1,n-1}(\mathbb{B}^n(x_0, 2r))$. Thus, $V(x)$ satisfies a Poincare inequality with respect to the euclidean metric. Thus,

$$\int_{\mathbb{B}^n(x_0,2r)} \eta(x)^{n-1} \left(\ln \ln \frac{1}{|F(x)|} \right)^{n-1} dx$$

$$\leq C\, R^{n-1} \int_{\mathbb{B}^n(x_0,2r)} |\nabla \left(\eta(x)^{n-1} \ln \ln \frac{1}{|F(x)|} \right) |^{n-1}\, dx.$$

Using the product rule on the right hand side of the above inequality, we obtain that

$$\int_{\mathbb{B}^n(x_0,2r)} |\nabla \left(\eta(x)^{n-1} \ln \ln \frac{1}{|F(x)|} \right) |^{n-1}\, dx$$

$$\leq 2^n \{ \int_{\mathbb{B}^n(x_0,2r)} |\nabla \eta(x)|^{n-1} \left(\ln \ln \frac{1}{|F(x)|} \right)^{n-1} dx$$

$$+ \int_{\mathbb{B}^n(x_0,2r)} \eta(x)^{n-1} |\nabla \left(\ln \ln \frac{1}{|F(x)|} \right) |^{n-1}\, dx \}. \tag{(2.3)}$$

By [MV], the second term on the right hand side of the above inequality is bounded by

$$\int_{\mathbb{B}^n(x_0,2r)} \eta(x)^{n-1} |\nabla \left(\ln \ln \frac{1}{|F(x)|} \right) |^{n-1}\, dx$$

$$\leq C \left(\int_{\mathbb{B}^n(x_0,2r)} |\nabla \eta(x)|^n\, K(x)^{n-1}\, dx \right)^{\frac{n-1}{n}} \left(\int_{\mathbb{B}^n(x_0,2r)} K(x)^{n-1}\, dx \right)^{\frac{1}{n}}.$$

Since by hypothesis, $K \in L^{n-1}$ and $\eta \in C_0^\infty(\mathbb{B}^n(x_0, 2r))$, the right hand side of the above inequality in uniformly bounded and thus, among the two terms on the right hand side of (2.3), the significant one is the first term, namely with possibly another constant C, we have that

$$\int_{\mathbb{B}^n(x_0,2r)} |\nabla \left(\eta(x)^{n-1} \ln \ln \frac{1}{|F(x)|} \right)|^{n-1} dx$$

$$\leq C r^{n-1} \int_{\mathbb{B}^n(x_0,2r)} |\nabla \eta(x)|^{n-1} \left(\ln \ln \frac{1}{|F(x)|} \right)^{n-1} dx$$

obtained the desired Poincare inequality that gives the inequality in Lemma 2.2.

To end section §2, we recall that for positive weights w, see [HK], and for a set $E \subset \mathbb{R}^n$, if we denote by

$$\Gamma(E) = \{\gamma \text{ rectifiable curves in } \mathbb{R}^n \text{ ending at a point in E}\}$$

we have that

$$(p, w) - \operatorname{cap}(E) = 0 \text{ if and only if } M_p^w(\Gamma(E)) = 0.$$

§3. Necessary conditions for the change of variable integral formula

We are going to need a change of variable formula for integrals. Let G be a domain in $\mathbb{R}^n$. A continuous mapping $F: G \to \mathbb{R}^n$ is said to satisfy Lusin's condition (N) if $|F(A)| = 0$ whenever $A \subset G$ and $|A| = 0$, where $|A|$ denotes the n-dimensional Lebesgue measure of A. Several necessary conditions have been found for a mapping F to satisfy Lusin's condition (N). For our purposes we will need the following result on that direction which can be found in [MZ].

Theorem. *Let F be a mapping in the Sobolev space $W^{1,n}(\Omega; \mathbb{R}^n)$ which is continuous and such that $J_F(x) > 0$ almost everywhere in Ω, then F satisfies condition (N) on Ω.*

We will use this result in this paper, since our result is local in natureand we are dealing with mappings of finite dilatation which are continuous. The condition $J_F > 0$ a.e. can be assumed without loss of generality from the sense preserving hypothesis on the mapping F, i.e. $J_F(x) \geq 0$ a.e. since otherwise F will be identically constant on a set of positive measure and locally we are assuming that our amppings are non constant.

More precisely, let us suppose that $J_F \geq 0$ a.e. does not imply that $J_F > 0$ a.e. Then, there exists an open set U in Ω with $|U| > 0$ such that for any $x \in U$ we have that $J_F(x) = 0$. We know that $1 \leq K(x) = \frac{|DF(x)|^n}{J_F(x)} < \infty$ a.e in Ω. Therefore, $|DF(x)| = 0$ a.e. in $U \subset \Omega$, which implies that F will be constant in an open subset of U of positive measure.

The following result can be found in [MZ].

Theorem 2.3. *Let $F \in W^{1,n}_{loc}(\Omega;\mathbb{R}^n)$ satisfy the Lusin's condition (N) locally in Ω. Then we have*

$$\int_\Omega \mu(F(x))\, J_F(x)\, dx = \int_{F(\Omega)} \mu(y)\, N(f, y; \Omega)\, dy,$$

where $N(F, y; \Omega)$ is the multiplicity function corresponding to F defined as the number (possibly infinity) of points in $F^{-1}(y) \cap \Omega$.

§4. Boundedness of $N(F, y; \Omega)$

In this section we will consider different conditions on the multiplicity function $N(F, y; \Omega)$ of the mapping F which will lead to show that the mapping is discrete and open.

First, we will talk about the Brouwer degree of the mapping F. The Brouwer degree $d(f, \Omega, p)$ of F with respect to Ω at a point $p \in \mathbb{R}^n \setminus F(\partial\Omega)$ is a well-defined integer depending only on the bounday values of the mapping F. It is well known, see [B], that if F is smooth (C^1), the topological degree of a mapping F $f\colon \Omega \to \mathbb{R}^n$ can be defined in a connected component V of $\mathbb{R}\setminus F(\partial\Omega)$ by

$$deg(F, \Omega, V) = \int_\Omega \rho(F(x))\, J_F(x)\, dx$$

where $\rho \in C^\infty_0(\mathbb{R}^n)$, such that its support is contained in V and satisfying the condition $\int_{\mathbb{R}^n} \rho(y)\, dy = 1$. Our next goal is to show that this is still true for mappings F continuous in $\bar{\Omega}$ and belonging to the Sobolev space $W^{1,n}_{loc}(\Omega)$.

Proof. Let V be a connected component of $\mathbb{R}^n \setminus F(\partial\Omega)$. It is clear that $d(F, \Omega, p)$ is independent of $p \in V$. We denote this common values for all $p \in V$ by $d(F, \Omega, V)$. Without loss of generality we can assume that $p = (0, 0, \dots, 0)$. Let A be a subset of Ω containing 0 with $A \subset \bar{A} \subset \Omega$, such that the support of rho lies in a connected component of $\mathbb{R}^n \setminus F(\bar{\Omega} \setminus A)$. Now we smooth tha mapping F on the set A. Let $\epsilon < \operatorname{dist}(A, \partial\Omega)$ and define the mollifiers, $g_\epsilon(x) = \frac{1}{\epsilon^n} g(\frac{x}{\epsilon})$ for any $x \in \mathbb{R}^n$ and $g \in C^\infty_0(\mathbb{R}^n)$ such that the suport of g is contained in the unit ball $\mathbb{B}^n(0,1)$ of $\mathbb{R}^n$ and $\int \mathbb{R}^n g(x)\, dx = 1$. let us define $F_\epsilon(x)$ as the convolution of the function g_ϵ with the mapping F as follows

$$F_\epsilon(x) = \int_{\mathbb{R}^n} g_\epsilon(x - y)\, F(y)\, dy,$$

for any $x \in A$. Since the convolution of two functions satisfy the commutative property, we obtain that

$$F_\epsilon(x) = \int_{\mathbb{R}^n} F(x - y)\, \frac{1}{\epsilon^n}\, g(\frac{x}{\epsilon})\, dy.$$

By performing the change of variable $\tilde{y} = \frac{y}{\epsilon}$ we obtain the following formula for F_ϵ,

$$f_\epsilon(x) = \int_{\mathbb{R}^n} F(x - \epsilon\tilde{y})\, g(\tilde{y})\, d\tilde{y}.$$

Thus, F_ϵ, the convolution of the mapping F with the function g_ϵ, belongs to $C(\bar{A}) \cap W^{1,n}(A)$. We will show that F_ϵ converges uniformly on compact subsets of A to the mapping F. For this, observe that

$$|F_\epsilon(x) - F(x)| = |\int_{\mathbb{R}^n} F(x + \epsilon y)\, g(y)\, dy - \int_{\mathbb{R}^n} F(x)\, g(y)\, dy|,$$

where to simplify the notation, we have replaced $-\epsilon$ by ϵ and $\tilde{y}$ by y. Hence, we have that

$$|F_\epsilon(x) - F(x)| = |\int_{\mathbb{R}^n} (F(x + \epsilon y) - F(x))\, g(y)\, dy|,$$

obtaining that

$$|F_\epsilon(x) - F(x)| \leq \int_{\mathbb{R}^n} |(F(x + \epsilon y) - F(x))|\, |g(y)|\, dy.$$

Let us denote by $G_\epsilon(x) = F(x + \epsilon\, y)$ for $x \in \Omega$. Then, it is trivial to see that the family of mappings $\{G_\epsilon\}_\epsilon$ converges uniformly to the mapping F in Ω as ϵ approaches 0. Thus, letting $\epsilon \to 0$, $|F_\epsilon(x) - F(x)| \to 0$ uniformly, which implies that F_ϵ converges to F uniformly on compact subsets of A as $\epsilon \to 0$. It is also clear that for ϵ small enough, the support of ρ lies in a connected component of $\mathbb{R}^n \setminus F_\epsilon(\partial A)$. Thus, since F_ϵ is a smooth mapping, we have the follwoing formula to compute its Browuwer degree,

$$\deg(F_\epsilon, \Omega, 0) = \int_{\mathbb{R}^n} \rho(F_\epsilon(x))\, J_{F_\epsilon}(x)\, dx,$$

letting $\epsilon \to 0$ we obtain the desired formula,

$$\deg(F, \Omega, 0) = \int_{\mathbb{R}^n} \rho(F(x))\, J_F(x)\, dx, \tag{(2.4)}$$

making use of the fact that $\rho(F_\epsilon(x))\, J_{F_\epsilon}(x) \to \rho(F(x))\, J_F(x)$ uniformly as $\epsilon \to 0$.

We have already established the validity in our case, of the change of variable formula,

$$\int_\Omega \mu(F(x))\, J_F(x)\, dx = \int_{F(\Omega)} \mu(y)\, N(F, \Omega, y)\, dy.$$

If we let in the above formula, $\mu = 1$, we have that

$$\int_\Omega J_F(x)\, dx = \int_{F(\Omega)} N(F, \Omega, y)\, dy. \tag{(2.5)}$$

Let us construct now a family of functions ρ_r as follows. Let V be a connected component of $\mathbb{R}^n \setminus F(\partial\Omega)$ in $F(\Omega)$. Let $Theta_r$ be a continuous function in $\mathbb{R}^n$ satisfying that $\Theta_r(p) = 1$ for all $p \in V$ such that $\text{dist}(p, \partial V) \geq \frac{1}{r}$, with its support in V and $0 \leq \Theta_r(p) \leq 1$ for any $p \in \mathbb{R}^n$.

Let us define the functions ρ_r as follows,

$$\rho_r(p) = \frac{\Theta_r(p)}{\int_{\mathbb{R}^n} \Theta_r(p)\, dp}.$$

Then we have that,

$$\deg(F, \Omega, 0) = \int_{\mathbb{R}^n} \rho_r(F(x))\, J_F(x)\, dx.$$

By the definition of ρ_r we have that

$$\deg(F, \Omega, 0) = \frac{1}{\int_{\mathbb{R}^n} \Theta_r(p)\, dp} \int_{\mathbb{R}^n} \Theta_r(F(x))\, J_F(x)\, dx,$$

letting $r \to \infty$ in the above equality, we obtain that

$$\deg(F, \Omega, 0) = \frac{1}{|V|} \int_{F^{-1}(V)} J_F(x)\, dx.$$

Thus, we have that

$$|V| \deg(F, \Omega, 0) = \int_{F^{-1}(V)} J_F(x)\, dx. \tag{(2.6)}$$

Combining (2.5) amd (2.6) we obtain that

$$|V| \deg(F, \Omega, 0) = \int_V N(F, \Omega, y)\, dy,$$

which proves that a.e. in $y \in V$ we have that

$$N(F, \Omega, y) = \deg(F, \Omega, 0).$$

Let us consider an special case. Let us suppose that for our mapping $F\colon \Omega \to \mathbb{R}^n$ there exists another mapping $F_0\colon \Omega \to \mathbb{R}^n$ which is continuous in $\bar{\Omega}$ and one to one in Ω. Let us further assume that $F|_{\partial\Omega} = F_)|_{\partial\Omega}$. Then, it is not difficult to show that $N(F, \Omega, y) = 1$ a.e. in $F(\Omega)$, see [Ball].

It is important to either show that under our hypotheis of the mapping F being quasi-light, then the multiplicity function $N(F, \Omega, y)$ is bounded or to impose restrictions on this multiplicity function that assure us that the openness and discreteness conclusions of our results still hold. Let us consider two different scenarios that allow us to conclude that our multiplicity function is bounded.

1. If the mapping F under consideration is quasi-light, then $N(F, \Omega, y)$ is bounded (or locally bounded in $F(\Omega)$, which for our purposes is enough, since our result is local in nature).

2. By [FG], we have that if the mapping $F \in W^{1,n}(\mathbb{B}^n(x, R))$ and $J_F(x) > 0$ a.e. then F has a differential almost everywhere (in the classical sense) and there exists a $R_0 > 0$ such that for every $0 < r < R_0$ we have that

$$\deg(F, \mathbb{B}^n(x_0, r), y) = 1$$

for every $y \in C_r$ where C_r is the connected component of $\mathbb{R}^n \setminus F(\partial \mathbb{B}^n(x_0, r))$ containing $y_0 = F(x_0)$, see Lemma 6.5 in [FG].

§5. A related result for the general case

Since our result is local, it will be enough to concentrate in a neighborhood of the origin and then by means of a linear transformation the result will remain valid for any $b \in \mathbb{R}^n$. Let Ω be an open ball such that $F^{-1}(0) = E \subset \Omega$. Let Δ be the family of rectifiable curves in $F(\Omega)$ ending at the origin. Similarly, let Δ_1 be the family of rectifiable curves ending at a point in E and completely contained in Ω. Let $\Gamma(\Delta)$ be the family of admissible metrics for Δ, and similarly, let $\Gamma(\Delta_1)$ be the family of admissible metrics for Δ_1.

Let $\gamma \in \Delta_1$ be such that its components $\gamma(t) = \{(x_1(t), x_2(t), \dots, x_n(t)) : t \in [a, b]\}$, then it follows that $F(\gamma) \in \Delta_1$ and

$$F(\gamma(t)) = \{(y_1(t), y_2(t), \dots, y_n(t)) : y_i(t) = F_i(x_1(t), x_2(t), \dots, x_n(t))$$

$$, i = 1, 2, \dots, n \text{ and } t \in [a, b]\}.$$

Let

$$d\bar{s}(y) = \sqrt{\left(\frac{\partial y_1}{\partial t}\right)^2 + \dots + \left(\frac{\partial y_n}{\partial t}\right)^2}.$$

By the definition of γ_i, $i = 1, 2, \dots, n$ and using the chain rule for differentiation we have that

$$\frac{\partial y_i}{\partial t} = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} \frac{\partial x_j}{\partial t},$$

for any $i = 1, 2, \dots, n$. Thus, we obtain that

$$d\bar{s}(y) = \sqrt{\left[\left(\sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i} \dot{x}_i\right)^2 + \dots + \left(\sum_{i=1}^{n} \frac{\partial F_n}{\partial x_i} \dot{x}_i\right)^2\right]},$$

where $\dot{x}_i$ stands for $\frac{dx_i}{dt}$.Applying Schwartz's inequality we obtain that

$$d\bar{s}(y) \leq \sqrt{\left[\sum_{i=1}^{n}\left(\frac{\partial F_1}{\partial x_i}\right)^2 \sum_{i=1}^{n}(\dot{x}_i)^2 + \ldots + \sum_{i=1}^{n}\left(\frac{\partial F_n}{\partial x_i}\right)^2 \sum_{i=1}^{n}(\dot{x}_i)^2\right]}.$$

Using the fact that $ds(x) = \sqrt{\sum_{i=1}^{n}(\dot{x}_i)^2}$ we obtain that

$$d\bar{s}(y) \leq \sqrt{\left[\sum_{i=1}^{n}\left(\frac{\partial F_1}{\partial x_i}\right)^2 (ds(x))^2 + \ldots + \sum_{i=1}^{n}\left(\frac{\partial F_n}{\partial x_i}\right)^2 (ds(x))^2\right]}$$

$$= ds(x)\sqrt{\left[\sum_{i=1}^{n}\left(\frac{\partial F_1}{\partial x_i}\right)^2 + \ldots + \sum_{i=1}^{n}\left(\frac{\partial F_n}{\partial x_i}\right)^2\right]}.$$

We also have that,

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\right)^2 \leq |DF(x)|^2$$

for every $j = 1, 2, \ldots, n$, which implies that

$$d\bar{s}(y) \leq \sqrt{n\,|DF(x)|^2}\, ds(x) = \sqrt{n}\,|DF(x)|\, ds(x).$$

We will use this inequality to show that if ρ is an admissible metric for the family of curves Δ then the metric defined ny $\sqrt{n}\,(\rho \circ F)(x)\,|DF(x)|$ is an admissible metric for the family of curves Δ_1. For this, let $\gamma \in \Delta_1$ then we have that $\hat{\gamma} = F(\gamma) \in \Delta$. Since ρ is an admissible metric for δ, we have that $1 \leq \int_{\hat{\gamma}} \rho(y)\, d\bar{s}(y)$ which implies that

$$1 \leq \sqrt{n}\int_{\gamma} \rho(F(x))\,|DF(x)|\, ds(x).$$

Since γ was any arbitrary curve in Δ_1, we have that the metric $\sqrt{n}\,\rho(F(x))\,|DF(x)|$ is admissible for the family Δ_1, that is, we have concluded that

$$\sqrt{n}\,\rho(F(x))\,|DF(x)| \in \Gamma(\Delta_1)$$

whenever $\rho \in \Gamma(\Delta)$.

Let us recall the definition of weighted module of order p when the weight function is identically equal to 1. We will denote by M_p. It follows from the definition that

$$M_p(\Delta_1) \leq \sqrt{n^p}\int_{\Omega} (\rho(F(x)))^p\,|DF(x)|^p\, dx.$$

Multiplying and dividing the integrand on the right hand side of the above inequality by $K(x)^{\frac{p}{n}}$ we obtain that

$$M_p(\Delta_1) \leq \sqrt{n^p}\int_{\Omega} (\rho(F(x)))^p\,|DF(x)|^p\, K(x)^{\frac{p}{n}}\,\frac{1}{K(x)^{\frac{p}{n}}}\, dx,$$

applying Hölder's inequality we have that

$$M_p(\Delta_1) \le C \left(\int_\Omega (\rho(F(x)))^n \frac{|DF(x)|^n}{K(x)} \, dx \right)^{\frac{p}{n}} \left(\int_\Omega \left(K(x)^{\frac{p}{n}} \right)^{\frac{n}{n-p}} dx \right)^{\frac{n-p}{n}},$$

where C is a constant depending only on n and p. Now, since $J_F(x) = \frac{|DF(x)|^n}{K(x)}$ we have that

$$((5.2)) \qquad M_p(\Delta_1) \le C \left(\int_\Omega (\rho(F(x)))^n J_F \, dx \right)^{\frac{p}{n}} \left(\int_\Omega K(x)^{\frac{p}{n-p}} \, dx \right)^{\frac{n-p}{n}}.$$

We are going to use the change of variable formula in the first integral on the right hand side of the above inequality. Thus, we have

$$\int_\Omega (\rho(F(x)))^n J_F \, dx = \int_{F(\Omega)} (\rho(y))^n N(F, \Omega, y) \, dy.$$

Now we will study different conditions on the growth of the multiplicity function $N(F, \Omega, y)$ which will still guarantee that by choosing a convenient sequence of admissible metrics ρ_η in the above equality and then taking the limit, that right hand side goes to zero. From now on, C will denote possibly different constants independent of F, Ω and ρ. Summarizing, we have that

$$((5.3)) \qquad M_p(\Delta_1) \le C \left(\int_{F(\Omega)} (\rho(y))^n N(F, \Omega, y) \, dy \right)^{\frac{p}{n}} \left(\int_\Omega K(x)^{\frac{p}{n-p}} \, dx \right)^{\frac{n-p}{n}}.$$

AT this point, let us assume that $N(F, \Omega, y)$ is bounded. It is also clear that we can replace in (5.3) $N(F, \Omega, y)$ by $N(F, \Omega, r) = \sup_{y \in \partial \mathbb{B}^n(0,r)} N(F, \Omega, y)$, which is now a radial function of r. That is, we have the alternating formula to (5.3),
((5.4))

$$M_p(\Delta_1) \le C \left(\int_0^R \int_{\partial \mathbb{B}^n(0,r)} (\rho(y))^n N(F, \Omega, r) \, dS(r) \, dr \right)^{\frac{p}{n}} \left(\int_\Omega K(x)^{\frac{p}{n-p}} \, dx \right)^{\frac{n-p}{n}}.$$

Case 1. $N(F, \Omega, y)$ is bounded.

Then by 5.3 we have that,

$$M_p(\Delta_1) \le C \left(\int_{F(\Omega)} (\rho(y))^n \, dy \right)^{\frac{p}{n}} \left(\int_\Omega K(x)^{\frac{p}{n-p}} \, dx \right)^{\frac{n-p}{n}}.$$

Let $p = n - 1 + \epsilon > n - 1$ and observe that

$$\frac{p}{n-p} = \frac{n-1+\epsilon}{1-\epsilon} = n - 1 + \frac{n\epsilon}{1-\epsilon} = n - 1 + \delta,$$

where $\delta = \frac{n\epsilon}{1-\epsilon}$. hence we have that

$$((2.6)) \qquad M_{n-1+\epsilon}(\Delta_1) \leq C \left(\int_{F(\Omega)} (\rho(y))^n \, dy \right)^{\frac{n-1+\epsilon}{n}} \left(\int_{\Omega} K(x)^{n-1+\delta} \, dx \right)^{\frac{1-\epsilon}{n}}.$$

Taking infimums in the above inequality we obtain that

$$((2.7)) \qquad M_{n-1+\epsilon}(\Delta_1) \leq C \left(M_n(\Delta)\right)^{\frac{n-1+\epsilon}{n}} \left(\int_{\Omega} K(x)^{n-1+\delta} \, dx \right)^{\frac{1-\epsilon}{n}}.$$

Next, we are going to show that $M_n(\Delta)$ is equal to zero. For this, we will use Lemma II.2.1 with the weight w identically equal to one, whic obviously belongs to the Muckenphout class A_n. The Lemma states that $M_n(\Delta) = 0$ if and only if

$$\int_{|x|<1} |x|^{(1-n)\frac{n}{n-1}} \, dx = \infty.$$

Using spherical coordinates in $\mathbb{R}^n$ it is trivial to show that the above integral becomes a divergent improper integral and we are done.

We would like to remind here, see [MV], that the reason to have an ϵ in our argument is because a classical result states that if $M_p(\delta_1) = 0$ then the Hausdorff diemnsion of E is less than or equal to $n-p$. In our situation, we want to ensure that $M_{n-1+\epsilon}(\Delta_1) = 0$, thus according to this classical result, Hausdorff dimension $(E) \leq 1 - \epsilon < 1$ which implies the discreteness of the set E and now by Titus and Young [TY], the openness of F.

AT this point, we can not conclude yet that $M_{n-1+\epsilon}(\Delta_1) = 0$ since we only know that $K(x) \in L^{n-1}_{loc}(\Omega)$ and the integral that appears on the right hand side of (2.7) is $\int_\Omega K(x)^{n-1+\delta} \, dx$. Since we can assume without loss of generality since our results are local, that $L^{n-1+\delta}_{loc}(\Omega) \subset L^{n-1}_{loc}(\Omega)$. At this point, and since the case $K(x) \in L^{n-1+\delta}_{loc}(\Omega)$ was completely settled by [MV], we can assume that $K(x) \in L^{n-1}_{loc}(\Omega) \setminus L^{n-1+\delta}_{loc}(\Omega)$. Thus, we are facing on the right hand side of (2.7) a product of the form zero times infinity. Our goal now will be to find which condition is needed for that indeterminate product to be zero. For this, let us define the following metric, $\rho = |\nabla \left(\ln \frac{1}{|y|} \right)^{\delta} |$ with δ positive and strictly less than $1 - \frac{1}{n}$.

We want to show that, somehow we can use these metrics to obtain an estimate of the modulus $M_n(\Delta)$. For this, we observe that Δ is the union of $\bigcup_{\eta>0} \Delta_\eta$, where Δ_η denotes the family of rectifiable curves in $\mathbb{B}^n(0, \eta)$ joining the origin with a point on the boundary of $\mathbb{B}^n(0,\eta)$. It is immediate to show that the modulus of this family of curves is bigger than the modulus of the family of curves in $\mathbb{R}^n$ with one end point at the origin and another at a boundary point of $\mathbb{B}^n(0,\eta)$. And this as we will see later in our argument will be enough for our arguments, thus we can restrict ourselves to estimate the modulus of the former family.

It is also clear that for each $\gamma \in \Gamma(\Delta)$ we have that $\int_\gamma \rho\, ds = \infty \geq 1$. Thus, it is admissible for $M_n(\Delta)$ and thus so are $\rho_\epsilon = \epsilon\, |\nabla\Big(\ln \frac{1}{|y|}\Big)^\delta|$. Let us compute now $\int_{\mathbb{B}^n(0,1/2)} \rho_\epsilon^n(x)\, dx$. A straightforward computation gives us that

$$|\nabla\Big(\ln \frac{1}{|y|}\Big)^\delta| = \delta\, \Big(\ln \frac{1}{|y|}\Big)^{\delta-1} \frac{1}{|y|}.$$

Let us denote $|y| = r$, thus we have after passing to spherical coordinates in $\mathbb{R}^n$

$$\int_{\mathbb{B}^n(0,1/2)} \rho_\epsilon^n(x)\, dx = C\, \epsilon^n\, \delta^n \int_0^1 /2\Big(\ln \frac{1}{r}\Big)^{n(\delta-1)} \frac{1}{r}^n\, r^{n-1}\, dr,$$

where C is a constant independent of δ, r, and ϵ. Using the change of variable $u = \ln \frac{1}{r}$ the above integral is transformed to

$$\int_{\mathbb{B}^n(0,1/2)} \rho_\epsilon^n(x)\, dx = C\, \epsilon^n\, \delta^n \int_{\ln 2}^{\infty} \Big(u\Big)^{n(\delta-1)} du,$$

the improper integral above converges by our choice of δ, and thus we have that

$$\int_{\mathbb{B}^n(0,1/2)} \rho_\epsilon^n(x)\, dx = C\, \epsilon^n,$$

letting $\epsilon \to 0$ we have shown that $M_n(\Delta) = 0$.

This provides a proof of Heinonen and Koskela's result, see [HK] in the Archive for Rational Mechanics, that a quasi-light mapping $F \in W^{1,n}_{loc}(\Omega; \mathbb{R}^n)$ with dilatation $K(x) \in L^p_{loc}(\Omega)$ for some $p > n-1$ is discrete and open.

§6. The weight $w(y) = (\ln \ln \frac{1}{|y|})^n$ is an A_n Muckhenhoupt weight

In order to prove our main result, which improves our previous ones, we need to introduce weighted modulus and their corresponding weighted variational capacities, and show that the weight

$$w(y) = (\ln \ln \frac{1}{|y|})^n$$

defined in $0 < |y| < 1$, satisfies the Muckenhoupt A_n condition in Ω.

We first observe that without loss of generality we can assume that $F(\Omega) \subset \mathbb{B}^n(0,1)$, with $0 \in F(\Omega)$, so that w is defined in $F(\Omega)$. It is also enough on the weight condition, to take the supremum over balls centered at 0 and show that

$$\sup_{0<r_0<1}\left[\frac{1}{r_0^n}\int_0^{r_0}\left(\ln\ln\frac{1}{r}\right)^n r^{n-1}\,dr\right]\left[\frac{1}{r_0^n}\int_0^{r_0}\left(\ln\ln\frac{1}{r}\right)^{\frac{n}{1-n}} r^{n-1}\,dr\right]^{n-1},$$

is finite. In order to show that, let us start finding a bound for the second factor on the right hand side of the above expression. For simplicity, we will denote that right hand side by $A(r_0)$. Since $r \leq r_0$, we obtain the following inequality

$$\ln\ln\frac{1}{r_0} \leq \ln\ln\frac{1}{r},$$

exponentiating, using the negative exponent $\frac{n}{1-n}$, we obtain that

$$\left(\ln\ln\frac{1}{r_0}\right)^{\frac{n}{1-n}} \leq \left(\ln\ln\frac{1}{r}\right)^{\frac{n}{1-n}}.$$

Thus, we have that

$$A(r_0) = \left[\frac{1}{r_0^n}\int_0^{r_0}\left(\ln\ln\frac{1}{r}\right)^{\frac{n}{1-n}} r^{n-1}\,dr\right]^{n-1}$$

$$\leq \left[\frac{1}{r_0^n}\int_0^{r_0}\left(\ln\ln\frac{1}{r_0}\right)^{\frac{n}{1-n}} r_0^{n-1}\,dr\right]^{n-1},$$

since $\int_0^{r_0} dr = r_0$ we obtain that,

$$A(r_0) \leq \left[\frac{1}{r_0^n}\left(\ln\ln\frac{1}{r_0}\right)^{\frac{n}{1-n}} r_0^n\,dr\right]^{n-1} = \ln\left(\ln\frac{1}{r_0}\right)^{-n}.$$

We shall now compute the integral on the first factor on the right hand side of the A_n condition. Namely,

$$\int_0^{r_0}\left(\ln\ln\frac{1}{r}\right)^n r^{n-1}\,dr.$$

For this, we will use the following substitution, $u = \frac{1}{r}$, then we have that $r = \frac{1}{u}$ and $dr = -\frac{1}{u^2}\,du$. Substituting this in the above integral we obtain that

$$\int_{\frac{1}{r_0}}^{\infty}\left(\ln\ln u\right)^n \frac{1}{u^{n+1}}\,du.$$

We will perform another substitution, $T = \ln u$, then $u = e^T$ and $du = e^T\, dT$, and hence the above integral becomes

$$\int_{\ln \frac{1}{r_0}}^{\infty} \left(\ln T\right)^n \frac{1}{e^{T(n+1)}}\, e^T\, dT = \int_{\ln \frac{1}{r_0}}^{\infty} \left(\ln T\right)^n \frac{1}{e^{Tn}}\, dT.$$

We will compute the last improper integral using the method of integration by parts. Let $u = \left(\ln T\right)^n$ and $dv = e^{-nT}\, dT$. Then, $du = n\left(\ln T\right)^{n-1} \frac{1}{T}\, dT$ and $v = -\frac{e^{-nT}}{n}$. Thus, integrating by parts, we obtain that the above integral is equal to

$$\int_{\ln \frac{1}{r_0}}^{\infty} \left(\ln T\right)^n \frac{1}{e^{Tn}}\, dT = \left[-\left(\ln T\right)^n \frac{e^{-nT}}{n}\right]_{\ln \frac{1}{r_0}}^{\infty}$$

$$+\int_{\ln \frac{1}{r_0}}^{\infty} -e^{-nT} \left(\ln T\right)^{n-1} \frac{1}{T}\, dT.$$

We observe now that the second term on the right hand side of the above equality is majorized by the first term , so we can disregard that term. thus, we obtain that

$$\int_{\ln \frac{1}{r_0}}^{\infty} \left(\ln T\right)^n \frac{1}{e^{Tn}}\, dT \approx \frac{r_0^n}{n} \left(\ln\, \ln \frac{1}{r_0}\right)^n .$$

Putting both estimates together in the A_n condition we obtain that

$$\sup_{0<r_0<1} \left[\frac{1}{r_0^n} \int_0^{r_0} \left(\ln\, \ln \frac{1}{r}\right)^n r^{n-1}\, dr\right] \left[\frac{1}{r_0^n} \int_0^{r_0} \left(\ln\, \ln \frac{1}{r}\right)^{\frac{n}{1-n}} r^{n-1}\, dr\right]^{n-1}$$

$$\leq \sup \left[\frac{1}{r_0^n} \frac{r_0^n}{n} \left(\ln\, \ln \frac{1}{r_0}\right)^n \left(\ln\, \ln \frac{1}{r_0}\right)^{-n}\right] = \frac{1}{n} < \infty,$$

and this concludes our proof that our weight $w(y) = (\ln\, \ln \frac{1}{|y|})^n$ belongs to the Muckenhoupt class $A_n(\Omega)$.

§7. Weighted $(n-1)$ variational capacities

In this section we will move from unweighted modulus and variational capacities to the weighted versions of them. This will allow us to improve on our results on the openness and discreteness of mappings F with integrable dilatation. We will also use h-Hausdorff-measures in our arguments in this section and the next.

We shall start by recalling the fact that if ρ is an admissible metric for the family of curves Δ, then $\sqrt{n}\,\rho(F(x))\,|DF(x)|$ is an admissible metric for the family of curves Δ_1 as it was proved in section §5. Let $w_1(x)$ be a positive weight defined by

$$w_1(x) = \left(\ln \ln \frac{1}{|F(x|}\right)^{n-1}$$

in Ω. Since our result is local we can assume without loss of generality that $|F(x)| < 1$ in Ω. Then by the definition of weighted modulus we have that

$$M_p^{w_1}(\Delta_1) \leq \int_\Omega \sqrt{n}^p\,(\rho(F(x)))^p \left(\ln \ln \frac{1}{|F(x|}\right)^{n-1} |DF(x)|^p\,dx,$$

multiplying and dividing the integrand on the right hand side of the above inequality by $K(x)^{\frac{p}{n}}$, we have that

$$M_p^{w_1}(\Delta_1) \leq \int_\Omega \sqrt{n}^p\,(\rho(F(x)))^p \left(\ln \ln \frac{1}{|F(x|}\right)^{n-1} |DF(x)|^p\,K(x)^{\frac{p}{n}}\,\frac{1}{K(x)^{\frac{p}{n}}}\,dx.$$

Applying Hölder's inequality, we obtain that

$$M_p^{w_1}(\Delta_1) \leq \sqrt{n}^p \left[\int_\Omega (\rho(F(x)))^n \left(\ln \ln \frac{1}{|F(x|}\right)^{\frac{(n-1)n}{p}} \frac{|DF(x)|^n}{K(x)}\,dx\right]^{\frac{p}{n}}$$
$$\left[\int_\Omega K(x)^{\frac{p}{n-p}}\,dx\right]^{\frac{n-p}{n}}.$$

Since $\frac{|DF(x)|^n}{K(x)} = J_F(x) > 0$ a.e. and using the formula for the change of variables on the first integral on the right hand side of the above inequality, we obtain that

$$M_p^{w_1}(\Delta_1) \leq \sqrt{n}^p \left[\int_{F(\Omega)} (\rho(y))^n \left(\ln \ln \frac{1}{|y|}\right)^{\frac{(n-1)n}{p}} N(F,\Omega,y)\,dy\right]^{\frac{p}{n}}$$
$$\left[\int_\Omega K(x)^{\frac{p}{n-p}}\,dx\right]^{\frac{n-p}{n}},$$

let $p = n-1$ in the above inequality to obtain

$$M^{w_1}_{n-1}(\Delta_1) \leq \sqrt{n}^{n-1} \left[\int_{F(\Omega)} (\rho(y))^n \left(\ln \ln \frac{1}{|y|} \right)^n N(F, \Omega, y)\, dy \right]^{\frac{n-1}{n}}$$

$$\left[\int_{\Omega} K(x)^{n-1}\, dx \right]^{\frac{1}{n}}.$$

Next, we are going to examine the integral

$$\int_{F(\Omega)} (\rho(y))^n \left(\ln \ln \frac{1}{|y|} \right)^n N(F, \Omega, y)\, dy.$$

First, we are going to assume that our metrics ρ are radial, that is, $\rho(y) = \rho(r)$ where $|y| = r$, and thus by taking $N(F, \Omega, r) = \sup_{y \in \partial \mathbb{B}^n(0,r)} N(F, \Omega, y)$, the above integral is less than or equal to

$$C \int_0^{\frac{1}{2}} (\rho(r))^n \left(\ln \ln \frac{1}{|y|} \right)^n N(F, \Omega, r)\, dr.$$

We will consider first the case $K(x) \in L^{n-1}_{loc}(\Omega)$ and F quasi-light, that is the multiplicity function $N(F, \Omega, y)$ is essentially bounded in Ω.

In [Hencl and Maly], it is shown that if the mapping $F \in W^{1,p}_{loc}(\Omega; \mathbb{R}^n)$, with $p > n-1$, be a continuous mapping with finite distortion which satisfies that $J_F \in L^1_{loc}(\Omega)$ and the equality

$$\operatorname{div}((\Psi \circ F)\ \operatorname{adj} DF) = ((\operatorname{div} \Psi) \circ F)\, J_F$$

holds in the sense of distributions in Ω for each C^1-vector field Ψ on $\mathbb{R}^n$. Then for any Ω' relatively compact subset of Ω we have that

$$N(F, \Omega', y) = \deg(F, \Omega', y)$$

for a.e. $y \in \mathbb{R}^n \setminus F(\partial\Omega')$.

Observe that our mapping F satisfies these hypothesis. Then taking into consideration our Remark 2 at the end of section §4, we have that for F quasi-light, choosing a point $x_0 \in \Omega$ such that $F(x_0) = 0$ we can find an open set Ω' compactly contained in Ω including the connected component of $F^{-1}(0)$ containing x_0, such that $0 \notin F(\partial\Omega')$. Let $\rho > 0$ such that

$$\bar{\mathbb{B}}^n((0, \rho) \cap F(\partial\Omega') = \emptyset$$

and define Ω'' as the connected component of $\Omega' \cap F(\mathbb{B}^n((0,\rho))$ which contains x_0. Then

$$N(F,\Omega'',y) = \deg(F,\Omega'',y) = \deg(F,\Omega'',0)$$

if $y \in \mathbb{B}^n((0,\rho)$ and $N(F,\Omega'',y) = \deg(F,\Omega'',y) = 0$ if $y \notin \mathbb{B}^n((0,\rho)$. This shows that $N(F,\Omega'',\ )$ is essentially bounded in a neighborhood of 0.

This implies that we have

$$M_{n-1}^{w_1}(\Delta_1) \leq C\left[\int_{F(\Omega)} (\rho(y))^n \left(\ln\ln\frac{1}{|y|}\right)^n dy\right]^{\frac{n-1}{n}}$$

$$\left[\int_{\Omega} K(x)^{n-1}\, dx\right]^{\frac{1}{n}}.$$

Taking infimums over all the admissible metrics ρ for the family of curves Δ we have that

$$M_{n-1}^{w_1}(\Delta_1) \leq \sqrt{n}^{n-1}\left(M_n^w(\Delta)\right)^{\frac{n-1}{n}}\left[\int_{\Omega} K(x)^{n-1}\, dx\right]^{\frac{1}{n}},$$

where the positive weight w is defined by

$$w(y) = \left(\ln\ln\frac{1}{|y|}\right)^n.$$

Now, as in the previous section, we want to show that $M_n^w(\Delta) = 0$.

We have already shown in the previous sections that $w(y) = \left(\ln\ln\frac{1}{|y|}\right)^n \in A_n(\Omega)$. Thus, all we need to show according to Lemma 2.3 is that the improper integral

$$\int_{\mathbb{B}^n(0,\frac{1}{e^e})} |y|^{(1-n)\frac{n}{n-1}} \left(\ln\ln\frac{1}{|y|}\right)^{\frac{n}{1-n}} dy$$

diverges. Taking spherical coordinates in $\mathbb{R}^n$ the above integral becomes

$$\int_0^{\frac{1}{e^e}} r^{-1}\left(\ln\ln\frac{1}{r}\right)^{\frac{n}{1-n}} dr.$$

Let us use the substitution $u = \ln\frac{1}{r}$ then $du = -\frac{1}{r}\,dr$, and the integral becomes

$$\int_e^{\infty} (\ln u)^{\frac{n}{1-n}}\, du.$$

Since we have that $\ln u \leq u^\epsilon$ for any ϵ positive and u large enough, and thus

$$\int_e^\infty (\ln u)^{\frac{n}{1-n}}\, du \geq \int_e^\infty (u)^{\epsilon \frac{n}{1-n}}\, du$$

$$\lim_{a\to\infty} \left[\frac{u^{\epsilon \frac{n}{1-n}+1}}{\epsilon \frac{n}{1-n} + 1} \right]_e^a .$$

By choosing ϵ so that $\epsilon \frac{n}{1-n} + 1 > 0$, the above limit is equal to infinity. Thus, it is enough to choose $0 < \epsilon < \frac{n-1}{n}$ and according to Lemma 2.3 $M_n^w(\Delta) = 0$. Since $K(x) \in L_{loc}^{n-1}(\Omega)$, we have that $M_{n-1}^{w_1}(\Delta_1) = 0$, where

$$w_1(x) = \left(\ln \ln \frac{1}{|F(x)|} \right)^{n-1} .$$

Because of the relation between weighted p-modulus and variational weighted p-capacities, we have that $(n-1, w_1) - \mathrm{cap}(E) = 0$. It remains to show in section §8 that this implies that the one dimensional Hausdorff measure of E is zero and that will complete the proof of our result.

§8. Proof of the main result

We want to show that the one dimensional Hausdorff measure of E is zero. Let us denote the one dimensional Hausdorff measure of E by

$$\Lambda_1(E) = \lim_{\delta\to 0} \left[\inf\{ \sum_i r_i \colon E \subset \bigcup \mathbb{B}^n(x_i, r_i),\ 0 < r_i < \delta \} \right],$$

where the infimum is taken over all coverings of E by balls of radii less than δ. By Lemma 2.2 in section §2 we have that for the weight w_1

$$r^{-(n-1)} \int_{\mathbb{B}^n(x_0,r)} \left(\ln \ln \frac{1}{|F(x)|} \right)^{n-1} dx$$

$$\leq C\,((n-1) - w_1) - \mathrm{cap}(\mathbb{B}^n(x_0,r), \mathbb{B}^n(x_0,2r)),$$

where C is a constant independent of x_0 and r.

Let $\mathbb{B}^n(0,\eta)$ and $\Omega_\eta = F^{-1}(\mathbb{B}^n(0,\eta))$. Now, consider a ring $\mathbb{B}^n(x_0,2r) \setminus \mathbb{B}^n(x_0,r)\emptyset$ completely contained in Ω_η centered at $x_0 \in E$. Observe that $|F(x)| \leq \eta$ for all $x \in \Omega_\eta$. Therefore, we have the following inequality

$$r^{-(n-1)} \int_{\mathbb{B}^n(x_0,r)} \left(\ln \ln \frac{1}{\eta} \right)^{n-1} dx = C \left(\ln \ln \frac{1}{\eta} \right)^{n-1} r$$

$$\leq r^{-(n-1)} \int_{\mathbb{B}^n(x_0,r)} \left(\ln \ln \frac{1}{|F(x)|}\right)^{n-1} dx.$$

Now, let us define $h(x_0, r)$ as follows

$$h(x_0, r) = r^{-(n-1)} \int_{\mathbb{B}^n(x_0,r)} \left(\ln \ln \frac{1}{|F(x)|}\right)^{n-1} dx,$$

it is immediate to see that

$$C\left(\ln \ln \frac{1}{\eta}\right)^{n-1} \leq \frac{h(x_0, r)}{r}.$$

Also observe that when $\eta \to 0$ we have that $x_0 \to E$ and $r \to 0$. Thus, we have that $\lim_{r\to 0,\, x_0 \to E} \frac{h(x_0,r)}{r}$, which implies that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $r < \epsilon\, h(x_0, r)$ whenever $r < \delta$ and x_0 is close enough to E. Let now $\{\mathbb{B}^n(x_i, r_i) \colon x_i \in E,\, 0 \leq r_i < \delta\}$ be a covering of the set E. If we define by

$$\Lambda_1^\delta(E) = \inf\{\sum_i r_i \colon E \subset \bigcup \mathbb{B}^n(x_i, r_i),\, 0 < r_i < \delta\},$$

where without loss of generality we can assume that all the x_i's are in E, we have that $\Lambda_1(E) = \lim_{\delta\to 0} \Lambda_1^\delta(E)$. If we choose δ as above, we have that

$$\Lambda_1^\delta(E) \leq \sum_i r_i < \sum_i \epsilon\, h(x_i, r_i)$$

$$\leq \epsilon C \{\sum_i ((n-1) - w_1) - \operatorname{cap}(\mathbb{B}^n(x_i, r_i), \mathbb{B}^n(x_i, 2r_i))\}.$$

We already know by the previous section that $((n-1) - w_1) - \operatorname{cap}(E) = 0$. Hence, by the definition of the weighted variational capacity and using rings to cover E instead of balls (observe that we can always assume that both rings and balls are centered at points of E), we have that for any $\tilde{\epsilon} > 0$ we can find a covering of E by rings such that $E \subset \bigcup_i (\mathbb{B}^n(x_i, r_i) \setminus \mathbb{B}^n(x_i, 2r_i))$ and

$$\sum_i ((n-1) - w_1) - \operatorname{cap}(\mathbb{B}^n(x_i, r_i), \mathbb{B}^n(x_i, 2r_i)) \leq ((n-1), w_1) - \operatorname{cap}(E) + \tilde{\epsilon}.$$

combining the above inequalities we have that

$$\Lambda_1^\delta(E) \leq C\epsilon \left[((n-1), w_1) - \operatorname{cap}(E) + \tilde{\epsilon}\right]$$

and since both ϵ and $\tilde{\epsilon}$ are arbitrary, letting $\delta \to 0$ we obtain that $\Lambda_1(E) = 0$. In particular $F^{-1}\{0\} = E$ can not contain a segment and thus, it is titally disconnected. replacing $F(x)$ by $F(x) - b$ in the above argument it follows that for any b, $F^{-1}\{b\}$ is totally disconnected. The mapping F is therefore an orientation preserving light mapping and it follows from a theorem of Titus and Young, see [TY], that the mapping F is open and discrete.

Now, we will consider the more general case in which the multiplicity function $N(F,\Omega,y)$ is not necessarily essentially bounded (quasi-light). In this case we have the inequality for the modulus

$$M^{w_1}_{n-1}(\Delta_1) \leq C\left[\int_{F(\Omega)} (\rho(y))^n \left(\ln\ln\frac{1}{|y|}\right)^n N(F,\Omega,y)\,dy\right]^{\frac{n-1}{n}}$$

$$\left[\int_{\Omega} K(x)^{n-1}\,dx\right]^{\frac{1}{n}}.$$

Our goal now will be to find which condition is necessary on $N(F,\Omega,y)$ which still guarantees that the first factor on the right hand side of the above inequality goes to zero, since that will imply that $M^{w_1}_{n-1}(\Delta_1) = 0$ due to the fact that $K(x) \in L^{n-1}_{loc}(\Omega)$. For this, let us define the following metric as in section §5, $\rho = |\nabla\left(\ln\frac{1}{|y|}\right)^{\delta}|$ with δ positive and strictly less than $1-\frac{1}{n}$.

We want to show that, somehow we can use these metrics to obtain an estimate of the modulus $M^{w_1}_{n-1}(\Delta_1)$.

It is also clear that for each $\gamma \in \Gamma(\Delta)$ we have that $\int_\gamma \rho\,ds = \infty \geq 1$. Thus, it is admissible for the family of curves Δ and thus so are $\rho_\epsilon = \epsilon|\nabla\left(\ln\frac{1}{|y|}\right)^{\delta}|$.

Without loss of generallity we can assume that $F(\Omega) \subset \mathbb{B}^n(0,\frac{1}{2})$. Let us compute now

$$\int_{\mathbb{B}^n(0,\frac{1}{2})} (\rho_\epsilon(y))^n \left(\ln\ln\frac{1}{|y|}\right)^n N(F,\Omega,y)\,dy.$$

Passing to spherical coordinates and considering

$$N(F,\Omega,r) = \sup_{y\in\partial\mathbb{B}^n(0,r)} N(F,\Omega,y),$$

the above integral is less than or equal to

$$C\int_0^{\frac{1}{2}} (\rho_\epsilon(y))^n \left(\ln\ln\frac{1}{|y|}\right)^n N(F,\Omega,r)\,dr$$

$$= C\int_0^{\frac{1}{2}} \epsilon^n\,\delta^n \left(\ln\frac{1}{r}\right)^{n(\delta-1)} \frac{1}{r}^n\, r^{n-1} \left(\ln\ln\frac{1}{r}\right)^n N(F,\Omega,r)\,dr$$

where C is a constant independent of δ, r, and ϵ. Using the change of variable $u = \ln \frac{1}{r}$ the above integral is transformed to

$$C\, \epsilon^n\, \delta^n \int_{\ln 2}^{\infty} \Big(u\Big)^{n(\delta-1)} (\ln u)^n\, N(F,\Omega,u)\, du.$$

The fact that $0 < \delta < 1 - \frac{1}{n}$ implies that $n(1-\delta) > 1$ thus $\Big(u\Big)^{n(\delta-1)} (\ln u)^n$ is integrable, which allows for the multiplicty function $N(F,\Omega,u)$ to be unbounded and yet the above improper integral to be convergent. Hence, by letting ϵ go to zero we will show that $M_{n-1}^{w_1}(\Delta_1) = 0$ and the same argument we used above will conclude that the mapping F is discrete and open. Hence we have proved the following result

Theorem 8.1. *Let $F \in W_{loc}^{1,n}(\Omega;\mathbb{R}^n)$ be a nonconstant mapping whose dilatation $K(x)$ is in $L_{loc}^{n-1}(\Omega)$. Let $N(F,\Omega,y)$ be its multiplicity function and we define $N(F,\Omega,r) = \sup_{y\in\partial\mathbb{B}^n(0,r)} N(F,\Omega,y)$. Then if we have that the improper integral*

$$\int_{\ln 2}^{\infty} \Big(u\Big)^{n(\delta-1)} (\ln u)^n\, N(F,\Omega,u)\, du$$

where $u = \ln \frac{1}{r}$ converges, then the mapping F is discrete and open.

Examples of unbounded $N(F,\Omega,r)$ for which the above improper integral converges are $N(F,\Omega,r) = \Big(\ln\, \ln \frac{1}{r}\Big)^p$ for any positive p and some positive r's. Observe that in our last result we are not assuming that the mapping F is quasi-light. So, our last result is on the direction of Iwaniec and Sverak's conjecture.

It will be interesting to study the behavior of $N(F,\Omega,r)$ as r tends to zero for mappings $F \in W_{loc}^{1,n}(\Omega;\mathbb{R}^n)$ whose dilatation $K(x)$ is in $L_{loc}^{n-1}(\Omega)$. If somehow we would be able to show that for any of those mappings, the improper integral

$$\int_{\ln 2}^{\infty} \Big(u\Big)^{n(\delta-1)} (\ln u)^n\, N(F,\Omega,u)\, du$$

converges where $u = \ln \frac{1}{r}$, this will prove the full Iwaniec and Sverak's conjecture.

Remark. In [Ball] it was conjectured that if a mapping $F \in W_{loc}^{1,n}(\Omega;\mathbb{R}^n)$ be a nonconstant sense preserving mapping whose dilatation $K(x)$ is in $L_{loc}^{n-1}(\Omega)$ and if $F_0\colon \bar{\Omega} \to \mathbb{R}^n$ be a continuous mapping in $\bar{\Omega}$ and one to one in Ω such that $F = F_0$ on $\partial\Omega$ then the mapping is discrete and open.

All we need to show is that for those mappings, $N(F,\Omega,y)$ is essentially bounded by one. This follows from Proposition 6 in [HM] and the fact shown in [Ball] that $\deg(F,\Omega,y) = 1$ for any $y \in F(\Omega)$ and $\deg(F,\Omega,y) = 0$ for any $y \in \mathbb{R}^n \setminus F(\bar{\Omega})$. Proposition 6 in [HM] shows that $N(F,\Omega',y) = \deg(F,\Omega',y)$ for a.e. $y \in \mathbb{R}^n \setminus F(\partial\Omega')$. Thus Ball's conjecture follows from our last theorem in this section §8.

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