Boundary Values of Quasiregular Mappings

The basic problem we would like to submit is the following:

Problem 1: Let $f: \mathbf{B}^n \to \mathbf{R}^n$, $n \geq 3$, be a bounded quasiregular mapping. Consider the set

$$E = \{\omega \in \partial \mathbf{B}^n \text{ such that } \lim_{t \to 1^-} f(t\omega) \text{ exists } \}.$$

Show that E is non empty.

Indeed we expect that $\partial \mathbf{B}^n \setminus E$ is small in some sense. We could ask the same question with radial limits replaced by nontangential limits, also called angular limits. It turns out that this is not really relevant since the existence of a radial limit implies the existence of the corresponding nontangential limit whenever the Harnack inequality is satisfied.

It seems that the current technology to prove Fatou theorems using pde's cannot deal with nonsmooth coefficients in the nonlinear case. Precisely what would be needed to treat the quasiregular case in higher dimensions. See **[MW]** and **[FGMS]**.

It is therefore natural to turn to more geometric methods of studying boundary values in the hope of advancing our understanding of problem 1. A simpler situation yet is obtained requiring that f satisfies, in addition, the condition

$$\int_{\mathbf{B}^n} |Df|^p \, dx < \infty, \tag{1}$$

for some $1 < p \leq n$. The case $p = n$ corresponds to considering mappings in

the Dirichlet class. In the two dimensional case $n = 2$ a classical theorem of Beurling **[B]** states that a harmonic function u on the unit disk $\mathbf{D}$ with finite Dirichlet integral has radial limits

$$\lim_{r \to 1^-} u(rx)$$

for all $x \in \partial \mathbf{D} \setminus E$, where E is a set of (logarithmic) capacity zero. This theorem has been generalized in many directions. The function u does not have to be harmonic. It is enough for u to be minimally regular (in the class ACL^2). See Section V in Carleson's book **[C]** for a detailed discussion and references.

A generalization to higher dimensions was given by Miklyukov in **[M]** where it is proven that that a bounded quasiregular mapping $f\colon \mathbf{B}^n \to \mathbf{R}^n$ satisfying (1) has angular boundary values everywhere on $\partial \mathbf{B}^n$ with the possible exception of a set of p-capacity zero. In this setting the boundedness hypothesis is not natural. It is used by Miklyukov to show that f is normal, which is always true for $p = n$. See Vuorinen's book **[V]** for the definition of normality and the proof of this fact (p. 189).

Problem 2: Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a quasiregular mapping satisfying (1). For what values of p is f normal?

Normal up to p-capacity zero for $n-1 < p \leq n$. Yamashita not true in general. [MV].

Problem 3: Does Miklyukov theorem hold without the boundedness hypothesis? Yes, [MV], $n-1 < p \leq n$.

We have recently found geometric proofs, using moduli of curves families, of Beurling's and Miklyukov theorems that give a positive answer to problem **3** for $p = n$. We also conjecture that both problems (**2** and **3**) have affirmative answers for $n - 1 < p \leq n$.

References

[B] Beurling, A., Ensembles exceptionelles, Acta Math 72 (1940).

[C] Carleson, L., *Selected Problems in Exceptional Sets*, Van Nostrand Mathematical Studies #13.

[FGMS] Fabes, E., Garofalo, N., Marín-Malave, S. and Salsa, S., Potential Theory and Fatou theorems for some nonlinear elliptic operators, to appear in the Revista Iberoamericana de Matemáticas.

[M] Miklyukov, V., Boundary Property of n-dimensional mappings with Bounded Distortion, Matematicheskie Zametki, Vol. 11(1972), No. 2, pp. 159–164.

[MW] Manfredi, J. and Weitsman, A., On the Fatou Theorem for p-harmonic functions, Comm. in PDE 13(6), 1988, pp. 651–688.

[V] Vuorinen, M., *Conformal Geometry and Quasiregular Mappings*, Lecture Notes 1319, Springer Verlag 1988.

Juan J. Manfredi, Dept. of Math., Univ. of Pittsburgh, Pittsburgh, PA 15260.
e-mail: manfredi@pittvms.bitnet
and
Enrique Villamor, Dept. of Math. Florida International Univ., Miami, FL. 33139.
e-mail: villamor@servax.bitnet